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AN

ELEMENTARY TREATISE

ON THE

DIFFERENTIAL CALCULUS.

By W. H. MILLER, M.A.

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AND PROFESSOR OF MINERALOGY IN THE UNIVERSITY OF CAMBRIDGE.

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DIFFERENTIAL CALCULUS.

SECTION I.

DIFFERENTIATION.

1. **When** any number of quantities are connected in such a manner, that the value of one of them depends upon the values of the remaining quantities, the former is called a function of the latter. When, however, a quantity depends upon others, some of which, in any particular case, are constant, and some variable, it is usually called a function of the variable quantities only, no notice being taken of those which are supposed to be constant.

Functions are said to be explicit, or implicit, according as they are, or are not, expressed directly in terms of the quantities upon which they depend.

Thus x^5, x^m, a^x, m and a being constant, are explicit functions of x. In the equation $y^3 - 3ayx + x^3 = 0$, a being constant, y is an implicit function of x. If $m^{\frac{1}{2}}t = a^{\frac{3}{2}}(u - e\sin u)$, and $\cos v\,(1 - e\sin u) = \cos u - e$, m, a, e, being constant, v is an implicit function of t. $\sin^{-1}\sqrt{(x^2 - y^2)}$ is an explicit function of x and y; and any root of an equation is an implicit function of the coefficients of the equation.

A function of any quantities is expressed, when the form of the function is unknown, by enclosing the quantities separated by commas, between brackets, and prefixing one of the letters f, F, ϕ, ψ,...Thus $f(x)$, $\psi(y)$, $F(x, y, z)$ denote respectively functions of x, of y and of x, y, z.

2. Let $f(x)$ be any function of x, and $f(x+h)$ the same function of $x+h$; then, if $f(x+h)$ be expanded in a series of ascending powers of h, the exponents of h will be positive, and the first exponent will be unity.

The series for $f(x+h)$ must reduce itself to $f(x)$ when h vanishes; consequently the exponents of h are all positive, (for if any of them were negative the series would become infinite when $h=0$) and the first term is $f(x)$.

We may therefore assume

$$f(x+h)=f(x)+f'(x)\,h^a+f''(x)\,h^b+\ldots\ldots(a)$$

where $f'(x)$, $f''(x)$,......are unknown functions of x, and a, b,...... positive quantities arranged in ascending order.

In the same manner we may assume

$$f'(x+h)=f'(x)+f'_{\prime}(x)\,h^{a_1}+f'_{\prime\prime}(x)\,h^{a_2}+\ldots$$

$$f''(x+h)=f''(x)+f''_{\prime}(x)\,h^{b_1}+f''_{\prime\prime}(x)\,h^{b_2}+\ldots$$

$$\ldots\ldots\ldots\ldots=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where a_1, a_2,...; b_1, b_2,......;...are arranged in ascending order, and are positive for the same reason that a, b,... are positive.

If we write $2h$ instead of h in (a) it becomes

$$f(x+2h)=f(x)+f'(x)\,2^a h^a+f''(x)\,2^b h^b+\ldots\ldots$$

And if we write $x+h$ instead of x in (a) it becomes

$$\begin{aligned}
f\{(x+h)+h\} &= f(x+h)+f'(x+h)\,h^a+f''(x+h)\,h^b+\ldots\\
&= f(x)+f'(x)\,h^a+f''(x)\,h^b+\ldots\\
&\qquad+\{f'(x)+f'_{\prime}(x)\,h^{a_1}+\ldots\}\,h^a\\
&\qquad\qquad+\{f''(x)+f''_{\prime}(x)\,h^{b_1}+\ldots\}\,h^b\\
&\qquad\qquad\qquad+\ldots\ldots\ldots\ldots\ldots\ldots\\
&= f(x)+2f'(x)\,h^a+2f''(x)\,h^b+\ldots\ldots
\end{aligned}$$

Equating the coefficients of h^a in the expansions of the identical expressions $f(x+2h)$ and $f\{(x+h)+h\}$, we obtain

$$f'(x)2^a = 2f'(x), \quad \therefore\ 2^a = 2, \quad \therefore\ a = 1\,;$$

$$\therefore\ f(x+h) = f(x) + f'(x)h + f''(x)h^b + \ldots\ldots\ldots (\beta).$$

It will be seen hereafter that cases may occur in which the expansion of $f(x+h)$, in the form given above, becomes impossible when x has a particular value.

3. The coefficient of h in the expansion of $f(x+h)$ is called "the differential coefficient of $f(x)$ with respect to x," and is denoted by the symbol $d_x f(x)$. The differential coefficient of $d_x f(x)$, or the coefficient of h in the expansion of $d_x f(x+h)$, is called the second differential coefficient of $f(x)$, and is denoted by the symbol $d_x^2 f(x)$, and so on.

Using this notation, equation (β) may be written thus

$$f(x+h) = f(x) + d_x f(x)h + f''(x)h^b + \ldots\ldots$$

The quantity x, which $f(x)$ contains, or depends upon, is termed the independant variable.

4. $d_x f(x) = \{[f(x+h) - f(x)] \div h\}_{h=0}$,

$\{[f(x+h) - f(x)] \div h\}_{h=0}$ denoting the value of $\{f(h+h) - f(x)\} \div h$ when h is indefinitely diminished.

$$f(x+h) = f(x) + d_x f(x)h + f''(x)h^b + \ldots (3);$$

$$\therefore\ d_x f(x) + f''(x)h^{b-1} + \ldots = \{f(x+h) - f(x)\} \div h:$$

and since $b-1, \ldots$ are all positive, $h^{b-1}, \ldots$ vanish when h vanishes, therefore

$$d_x f(x) = \{[f(x+h) - f(x)] \div h\}_{h=0}.$$

5. Let y be a function of x, and u a function of y; then

$$d_x u = d_y u\, d_x y.$$

When x becomes $x+h$, y will become

$$y + d_x y h + \dots (3) = y + k, \text{ if } d_x y h + \dots = k.$$

In like manner when y becomes $y+k$, u will become

$$u + d_y u k + \dots = u + d_y u\,(d_x y h + \dots) + \dots$$

And $d_x u =$ coefficient of h in the preceding expansion of the new value of $u = d_y u\, d_x y$.

If $u = x$, $d_x u = d_x x =$ coefficient of h in $x + h = 1$;

$$\therefore\ 1 = d_y x d_x y.$$

6. $d_x(au + c) = ad_x u$, u being a function of x, and a and c quantities independant of x.

When x becomes $x+h$, u becomes $u + d_x u h + \dots$ (3), $\therefore$ $au + c$ becomes $a(u + d_x u h + \dots) + c$, and $d_x(au+c)$ = coefficient of h in the expansion of the new value of $au + c$ $= ad_x u$.

Hence $d_x c = 0$, when c does not contain x.

7. $d_x(u_1 + u_2 + \dots + u_n) = d_x u_1 + d_x u_2 + \dots + d_x u_n$.

When x becomes $x+h$, u_1, u_2, ... u_n become

$$u_1 + d_x u_1 h + \dots,\ u_2 + d_x u_2 h + \dots, \dots\dots\ u_n + d_x u_n h + \dots$$

respectively, $\therefore$ $u_1 + u_2 + \dots + a_n$ becomes

$$u_1 + u_2 + \dots u_n + (d_x u_1 + d_x u_2 + \dots + d_x u_n)\,h + \dots,$$

and $d_x(u_1 + u_2 + \dots + u_n) =$ coefficient of h in the new value of $u_1 + u_2 + \dots + u_n$

$$= d_x u_1 + d_x u_2 + \dots\dots d_x u_x.$$

8. $d_x(uv) = ud_x v + vdu$.

When x becomes $x+h$, u and v become

$$u + d_x u h + \dots \text{ and } v + d_x v h + \dots,$$

$$\therefore\ uv \text{ becomes } (u + d_x u h + \ldots)(v + d_x v h + \ldots)$$
$$= uv + (u d_x v + v d_x u)\, h + \ldots,$$

and $d_x(uv)$ = coefficient of h in the new value of uv

$$= u d_x v + v d_x u.$$

Dividing both sides by uv, we obtain

$$\frac{1}{uv} d_x(uv) = \frac{1}{u} d_x u + \frac{1}{v} d_x v.$$

9. $$\frac{1}{u_1 u_2 \ldots u_n} d_x(u_1 u_2 \ldots u_n) = \frac{1}{u_1} d_x u_1 + \frac{1}{u_2} d_x u_2 + \ldots + \frac{1}{u_n} d_x u_n.$$

$$\frac{1}{u_1 u_2} d_x(u_1 u_2) = \frac{1}{u_1} d_x u_1 + \frac{1}{u_2} d_x u^2 \quad (8).$$

$$\frac{1}{u_1 u_2 u_3} d_x(u_1 u_2 u_3) = \frac{1}{u_1 u_2} d_x(u_1 u_2) + \frac{1}{u_3} d_x u_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots \quad + \ldots\ldots$$

$$\frac{1}{u_1 \ldots u_{n-1}} d_n(u_1 \ldots u_{n-1}) = \frac{1}{u_1 \ldots u_{n-2}} d_x(u_1 \ldots u_{n-2}) + \frac{1}{u_{n-1}} d_x u_{n-1}$$

$$\frac{1}{u_1 \ldots u_n} d_x(u_1 \ldots u_n) = \frac{1}{u_1 \ldots u_{n-1}} d_x(u_1 \ldots u_{n-1}) + \frac{1}{u_n} d_x u_n.$$

Adding up, and observing that every term, except the last, on the left hand side of the equation is destroyed by an equal term on the right, we obtain

$$\frac{1}{u_1 u_2 \ldots u_n} d_x(u_1 u_2 \ldots u_n) = \frac{1}{u_1} d_x u_1 + \frac{1}{u_2} d_x u_2 + \ldots + \frac{1}{u_n} d_x u_n.$$

10. $d_x u^n = n u^{n-1} d_x u.$

First, let n be a positive whole number,

and in (9) let $u_n = u_{n-1} = \ldots = u_2 = u_1 = u$, $\therefore\ u_1 u_2 \ldots u_n = u^n$,

$$\text{and } \frac{1}{u_1} d_x u_1 + \frac{1}{u_2} d_x u_2 + \ldots + \frac{1}{u_n} d_x u_n = \frac{n}{u} d_x u,$$

$$\therefore\ \frac{1}{u^n} d_x u^n = \frac{u}{n} d_x u, \quad \therefore\ d_x u^n = n u^{n-1} d_x u.$$

Next, let $n = \frac{r}{s}$, r and s being integers,

and let $u^{\frac{r}{s}} = v$, $\therefore\ v^s = u^r$, $\therefore\ s v^{s-1} d_x v = r u^{r-1} d_x u$,

$$d_x u^{\frac{r}{s}} = \tfrac{r}{s} u^{r-1} (u^{\frac{r}{s}})^{-(s-1)} d_x u = \tfrac{r}{s} u^{\frac{r}{s}-1} d_x u.$$

Lastly, let $n = -m$, a negative quantity, and let

$u^{-m} = v$, $\therefore\ u^m v = 1$, $\therefore\ u^m d_x v + v m u^{m-1} d_x u = 0$, (6), (9),

$$\therefore\ d_x u^{-m} = -\frac{u^{-m}}{u^m} m u^{m-1} d_x u = -m u^{-m-1} d_x u.$$

Therefore, $d_x u^n = n u^{n-1} d_x u$ in every case.

Hence $d_x \sqrt{(u)} = \dfrac{d_x u}{2\sqrt{(u)}}$, and $d_x \dfrac{1}{u} = -\dfrac{d_x u}{u^2}$.

11. If in (10) we make $u = x$, $d_x x^n = n x^{n-1} d_x x = n x^{n-1}$.

This may be proved independently by the following process, which is taken with slight alterations from "Swinburne's and Tylecote's true Development of the Binomial Theorem."

When n is a positive whole number

$$\frac{(1+z)^n - 1}{(1+z) - 1} = (1+z)^{n-1} + (1+z)^{n-2} + \ldots + (1+z)^2 + 1 + z + 1;$$

$$\therefore\ (1+z)^n = 1 + z\left\{(1+z)^{n-1} + (1+z)^{n-2} + \ldots + (1+z)^2 + 1 + z + 1\right\}$$

Similarly the first term of each of the n quantities

$$(1+z)^{n-1},\ (1+z)^{n-2}, \ldots (1+z)^2,\ (1+z),\ 1,$$

is 1; every other term is multiplied by some power of z;

$$\therefore\ (1+z)^n = 1 + z\,\{n + \text{terms involving } z, z^2, \ldots\}.$$
$$= 1 + nz + \text{terms involving } z^2, z^3, \ldots$$

When $n = -m$, a negative whole number,

$$\frac{(1+z)^{-m}-1}{(1+z)^{-1}-1} = (1+z)^{-m+1}+(1+z)^{-m+2}+\ldots+(1+z)^{-2}+(1+z)^{-1}+1;$$

$$\begin{aligned}\therefore\ (1+z)^{-m} &= 1 - z\left\{(1+z)^{-m}+(1+z)^{-m+1}+\ldots+(1+z)^{-1}\right\}\\ &= 1 - z\left\{m + \text{terms involving } z, z^2, \ldots\right\}\\ &= 1 - mz - \text{terms involving } z^2, z^3, \ldots\end{aligned}$$

When $n = \frac{r}{s}$, a positive or negative fraction, r and s being whole numbers, let

$$(1+z)^{\frac{r}{s}} = 1 + Az + \ldots$$

$$\therefore\ (1+z)^r = (1 + Az + \ldots)^s,$$

$$\therefore\ 1 + rz + \ldots = 1 + s(Az + \ldots) + \ldots$$

$$\therefore\ A = \frac{r}{s},\quad \therefore\ (1+z)^{\frac{r}{s}} = 1 + \frac{r}{s}z + \ldots$$

Hence, in every case,

$$(1+z)^n = 1 + nz + \text{terms in } z^2, z^3, \ldots$$

Let $z = \dfrac{h}{x}$, $\therefore\ \left(1+\dfrac{h}{x}\right)^n = 1 + n\dfrac{h}{x} + \ldots$

$$\therefore\ (x+h)^n = x^n + nx^{n-1}h + \ldots$$

And $d_x x^n =$ coefficient of h in the expansion of $(x+h)^n = nx^{n-1}$.

12. To find the successive differential coefficients of x^n.

$$d_x x^n = nx^{n-1}.$$

$$d_x^2 x^n = d_x n x^{n-1} = n(n-1)x^{n-2}.$$

$$d_x^2 x^n = d_x n(n-1)x^{n-2} = n(n-1)(n-2)x^{n-3}.$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$d_x^m x^n = n(n-1)(n-2)\dots(n-m)\,x^{n-m}.$$

$$\dots = \dots\dots\dots\dots\dots\dots\dots\dots$$

$$d_x^n x^n = n(n-1)(n-2)\dots 3.2.1.$$

13. To expand $(1+x)^n$ in a series of ascending powers of x.

It appears from (11) that we may assume

$$(1+x)^n = 1 + nx + Bx^b + Cx^c + Dx^d + \dots + Qx^q + Rx^r + \dots$$

where b, c, d, ... q, r, ... are arranged in ascending order.

Differentiating with respect to x, we get

$$n(1+x)^{n-1} = n + bBx^{b-1} + cCx^{c-1} + dDx^{d-1} + \dots + qQx^{q-1} + rRx^{r-1} +$$

Multiplying by $(1+x)$

$$\begin{aligned} n(1+x)^n = n &+ bBx^{b-1} + cCx^{c-1} + dDx^{d-1} + \dots + qQx^{q-1} + rRx^{r-1} + \\ &+ nx \quad + bBx^b \quad + cCx^c \quad + \dots\dots\dots\dots + qQx^q \quad + \end{aligned}$$

But

$$n(1+x)^n = n + n^2x + nBx^b + nCx^c + \dots\dots\dots\dots + nQx^q +$$

Therefore, equating the terms of these two expansions of $n(1+x)^n$ taken in order,

$$b-1=1,\quad B = \frac{n^2-n}{2} = n\frac{n-1}{2};$$

$$c-1=2,\quad C = B\frac{n-2}{3} = n\frac{n-1}{2}\,\frac{n-2}{3};$$

$$d-1=3,\quad D = C\frac{n-3}{4} = n\frac{n-1}{2}\,\frac{n-2}{3}\,\frac{n-3}{4};$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$r-1=q,\quad R = Q\frac{n-r+1}{r} = n\frac{n-1}{2}\,\frac{n-2}{3}\dots\dots\frac{n-r+1}{r};$$

$$\therefore\ (1+x)^n = 1 + nx + n\frac{n-1}{2}x^2 + n\frac{n-1}{2}\,\frac{n-2}{3}x^3 + \dots$$

$$\dots\dots + n\frac{n-1}{2}\,\frac{n-2}{3}\dots\dots\frac{n-r+1}{r}x^r + \dots\dots$$

14. To find the successive differential coefficients of uv.

$$d_x(uv) = u d_x v + d_x u\,.\,v.$$

$$d_x^2(uv) = u d_x^2 v + 2 d_x u d_x v + d_x^2 u\,.\,v.$$

$$d_x^3(uv) = u d_x^3 v + 3 d_x u d_x^2 v + 3 d_x^2 u d_x v + d_x^3 u\,.\,v.$$

$$\dots\dots = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The coefficients and number of differentiations of u, v, in $d_x^2(uv)$, $d_x^3(uv)$, are the same respectively as the coefficients and indices of u, v, in $(v+u)^2$, $(v+u)^3$. Let us endeavour to find whether this law holds generally. If

$$d_x^n(uv) = \dots\dots\dots + n\frac{n-1}{2}\dots\frac{n-r+2}{r-1}d_x^{r-1}u\,d_x^{n-r+1}v$$

$$+ n\frac{n-1}{2}\dots\frac{n-r+1}{r}d_x^r u\,d_x^{n-r}v + \dots$$

$$d_x^{n+1}(uv) = \dots\dots + n\frac{n-1}{2}\dots\frac{n-r+2}{r-1}d_x^r u\,d_x^{n-r+1}v$$

$$+ n\frac{n-1}{2}\dots\frac{n-r+1}{r}d_x^r u\,d_x^{n-r+1}v + \dots$$

$$= \dots\dots + (n+1)\frac{n}{2}\,.\,\frac{n-1}{3}\dots\frac{n-r+2}{r}d_x^r u\,d_x^{n-r+1}v + \dots$$

therefore, if the law is true for $d_x^n(uv)$, it is true for $d_x^{n+1}(uv)$. But it is true for $d_x^2(uv)$ and $d_x^3(uv)$, therefore it is true for $d_x^4(uv)$, $d_x^5(uv)$, ... $d_x^n(uv)$, ...

15. $d_x\left(\frac{u}{v}\right) = \frac{1}{v^2}(v\,d_xu - u\,d_xv)$.

$$\text{Let } \frac{u}{v} = w, \quad \therefore\ u = wv,$$

$$\therefore\ d_xu = w\,d_xv + v\,d_xw \quad (8) \quad = \frac{u}{v}\,d_x v + v\,d_x\left(\frac{u}{v}\right),$$

$$\therefore\ d_x\left(\frac{u}{v}\right) = \frac{1}{v^2}(v\,d_xu - u\,d_xv).$$

$$\text{Dividing by } \frac{u}{v},\ \frac{v}{u}\,d_x\frac{u}{v} = \frac{1}{u}\,d_xu - \frac{1}{v}\,d_xv.$$

16. $\dfrac{v_1v_2\dots v_n}{u_1u_2\dots u_m}\,d_x\dfrac{u_1u_2\dots u_m}{v_1v_2\dots v_n}$

$$= \frac{1}{u_1}\,d_xu_1 + \frac{1}{u_2}\,d_xu_2 + \dots + \frac{1}{u_m}d_xu_m.$$

$$-\left(\frac{1}{v_1}\,d_xv_1 + \frac{1}{v_2}\,d_xv_2 + \dots + \frac{1}{v_n}\,d_xv_n\right).$$

Let $u_1u_2\dots u_m = u$, $v_1v_2\dots v_n = v$,

$$\frac{v}{u}\,d_x\frac{u}{v} = \frac{1}{u}\,d_xu - \frac{1}{v}\,d_xv \quad (14),$$

$$\frac{1}{u}\,d_xu = \frac{1}{u_1}\,d_xu_1 + \frac{1}{u_2}\,d_xu_2 + \dots + \frac{1}{u_m}\,d_xu_m,$$

$$\frac{1}{v}\,d_xv = \frac{1}{v_1}\,d_xv_1 + \frac{1}{v_2}d_xv_2 + \dots + \frac{1}{v_n}d_xv_n, \quad (9)$$

$$\therefore\ \frac{v_1v_2\dots v_n}{u_1u_2\dots u_m}\,d_x\frac{u_1u_2\dots u_m}{v_1v_2\dots v_n} = \frac{1}{u_1}d_xu_1 + \frac{1}{u_2}\,d_xu_2 + \dots + \frac{1}{u_m}\,d_xu_m.$$

$$-\left(\frac{1}{v_1}d_xv_1 + \frac{1}{v_2}\,d_xv_2 + \dots + \frac{1}{v_n}\,d_xv_n\right).$$

17. $d_x a^x = A a^x$,

$$\text{where } A = \frac{a-1}{a} + \tfrac{1}{2}\left(\frac{a-1}{a}\right)^2 + \tfrac{1}{3}\left(\frac{a-1}{a}\right)^3 + \ldots\ldots$$

$$a^h = \left(\frac{1}{a}\right)^{-h} = \left(1 - \frac{a-1}{a}\right)^{-h}$$

$$= 1 + h\frac{a-1}{a} + h\frac{h+1}{2}\left(\frac{a-1}{a}\right)^2 + h\frac{h+1}{2}\,\frac{h+2}{3}\left(\frac{a-1}{a}\right)^3 + \ldots\ldots$$

$$= 1 + \left\{\frac{a-1}{a} + \tfrac{1}{2}\left(\frac{a-1}{a}\right)^2 + \tfrac{1}{3}\left(\frac{a-1}{a}\right)^3 + \ldots\right\}h + Bh^2 + Ch^3 + \ldots$$

(arranging the series according to powers of h)

$= 1 + Ah + Bh^2 + Ch^3 + \ldots$

$$a^{x+h} = a^x a^h = a^x(1 + Ah + Bh^2 + Ch^3 + \ldots):$$

and $d_x a^x$ = coefficient of h in the expansion of $a^{x+h} = A a^x$.

18. To obtain a series for A, which may be made to converge rapidly in every case.

$$a^h = \left(a^{\frac{1}{2^n}}\right)^{2^n h} = c^k \text{ (if } c = a^{\frac{1}{2^n}}, \text{ and } k = 2^n h)$$

$$= 1 + \left\{(1 - c^{-1}) + \tfrac{1}{2}(1 - c^{-1})^2 + \tfrac{1}{3}(1 - c^{-1})^3 + \ldots\right\}k + \ldots \quad (17)$$

$$= 1 + \left\{(1 - a^{-\frac{1}{2^n}}) + \tfrac{1}{2}(1 - a^{-\frac{1}{2^n}})^2 + \tfrac{1}{3}(1 - a^{-\frac{1}{2^n}})^3 + \ldots\quad\right\}2^n h + \ldots$$

$$\therefore A = \text{coeff}^{\text{t}} \text{ of } h = 2^n\left\{(1 - a^{-\frac{1}{2^n}}) + \tfrac{1}{2}(1 - a^{-\frac{1}{2^n}})^2 + \tfrac{1}{3}(1 - a^{-\frac{1}{2^n}})^3 + \ldots\right\}$$

$a^{\frac{1}{2^n}}$ is obtained by extracting the square root of a n times successively; and by increasing n we may make $1 - a^{-\frac{1}{2^n}}$ as small, and therefore the series as convergent, as we please.

19. To expand a^x in a series of ascending powers of x.

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \ldots \quad (16),$$

differentiating with respect to x, we get

$$Aa^x = A + 2Bx + 3Cx^2 + \ldots \text{But}$$

$$Aa^x = A + A^2x + ABx^2 + \ldots$$

Therefore, equating the coefficients of like powers of x, we obtain

$$B = \frac{A^2}{1.2}, \quad C = \frac{AB}{3} = \frac{A^3}{1.2.3}, \quad \&c. = \&c.$$

$$\therefore \ a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \ldots\ldots$$

20. $\mathrm{Log}_a x$, "the logarithm of x to the base a," is a quantity that satisfies the equation $a^{\log_a x} = x$.

Let E be the same function of ϵ that A is of a, and let $E = 1$;

$$\therefore \ \epsilon = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots\ldots = 2.71828182846\ldots$$

But $a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots;$

$$\therefore \ \epsilon = a^{\frac{1}{A}}, \ \therefore \ \epsilon^A = a = \epsilon^{\log_\epsilon a}, \ \therefore \ A = \log_\epsilon a.$$

$$\therefore \ a^x = 1 + (\log_\epsilon a)x + (\log_\epsilon a)^2\frac{x^2}{1.2} + (\log_\epsilon a)^3\frac{x^3}{1.2.3} + \ldots$$

And $\epsilon^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots\ldots\ldots$

$$\text{Since } E = 1, \ d_x\epsilon^x = E\epsilon^x = \epsilon^x.$$

Also $d_xa^x = Aa^x = (\log_\epsilon a)a^x$.

This might have been deduced at once from (17). For,

$$a^x = (\epsilon^{\log_\epsilon a})^x = \epsilon^{\log_\epsilon a . x} = \epsilon^y, \text{ if } y = \log_\epsilon a . x.$$

$$d_x a^x = d_x \epsilon^y = d_y \epsilon^y d_x y = \epsilon^y d_x y = \log_\epsilon a . a^x.$$

21. To find the successive differential coefficients of a^x.

$$d_x a^x = (\log_\epsilon a) a^x.$$

$$d_x^2 a^x = (\log_\epsilon a)^2 a^x.$$

$$d_x^3 a^x = (\log_\epsilon a)^3 a^x.$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$d_x^n a^x = (\log_\epsilon a)^n a^x.$$

Hence $d_x^n \epsilon^x = \epsilon^x$.

22. $d_x \log_a x = \dfrac{1}{\log_\epsilon a . x}$.

If $y = \log_a x$, $a^y = x$, $\therefore\ d_x a^y = d_x x$,

$$d_x a^y = d_y a^y d_x y = \log_\epsilon a . a^y d_x y = \log_\epsilon a . x \, d_x \log_a x,$$

and $d_x x = 1$, $\therefore\ d_x \log_a x = \dfrac{1}{\log_\epsilon a . x}$.

$$d_x \log_\epsilon x = \frac{1}{x}.$$

23. To find the successive differential coefficients of $\log_a x$.

$$d_x \log_a x = (\log_\epsilon a)^{-1} x^{-1}.$$

$$d_x^2 \log_a x = -(\log_\epsilon a)^{-1} . x^{-2}.$$

$$d_x^3 \log_a x = 1 . 2 . (\log_\epsilon a)^{-1} . x^{-3}.$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$d_x^n \log_a x = -1 . 2 . 3 \ldots\ldots (n-1)(-1)^n (\log_\epsilon a)^{-1} x^{-n}.$$

$$d_x^n \log_\epsilon x = -1 . 2 . 3 \ldots\ldots (n-1)(-1)^n x^{-n}.$$

24. $d_x u^z = u^z(z\dfrac{d_x u}{u} + \log_\epsilon u\, d_x z)$

$$\log_\epsilon u^z = z \log_\epsilon u,$$

$$d_x \log_\epsilon u^z = z d_x \log_\epsilon u + \log_\epsilon u\, d_x z, \text{ or}$$

$$\frac{d_x u^z}{u^z} = z\frac{d_x u}{u} + \log_\epsilon u\, d_x z;$$

$$\therefore\ d_x u^z = u^z(z\frac{d_x u}{u} + \log_\epsilon u\, d_x z).$$

In the same manner we may differentiate u^{z^y} and similar exponential functions.

25. To expand $\log_\epsilon(1+x)$ in a series of ascending powers of x.

Since $\log_\epsilon 1 = 0$, the series must vanish when $x = 0$, $\therefore$ a positive power of x enters every term of it as a multiplier. Consequently we may assume

$$\log_\epsilon(1 + x) = Ax^a + Bx^b + Cx^c + Dx^d + \ldots$$

where $a, b, c, d, \ldots$ are arranged in ascending order.

Differentiate, and we get

$$\frac{1}{1+x} = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} + dDx^{d-1} + \ldots$$

But $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$

Therefore, equating the terms taken in order,

$$a - 1 = 0,\ A = 1;\ b - 1 = 1,\ B = -\tfrac{1}{2};\ c - 1 = 2,\ C = \tfrac{1}{3};$$
$$d - 1 = 3,\ D = -\tfrac{1}{4};\ \text{\&c.} = \text{\&c.};$$

$$\therefore\ \log_\epsilon(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

This series is not sufficiently convergent to be employed in the calculation of logarithms, except when x is a small fraction.

26. To express $\log_\epsilon (1 + x)$ in a converging series.

$$\log_\epsilon (1 + x) = \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$$\log_\epsilon (1 - x) = - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

$$\log_\epsilon \frac{1 + x}{1 - x} = \log_\epsilon (1 + x) - \log_\epsilon (1 - x);$$

$$\therefore \log_\epsilon \frac{1 + x}{1 - x} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right). \qquad (\alpha)$$

$$1 + x = \left(1 + \frac{x}{x + 2}\right) \div \left(1 - \frac{x}{x + 2}\right);$$

$\therefore$ writing $\dfrac{x}{x + 2}$ instead of x in (α),

$$\log_\epsilon (1 + x) = 2 \left\{\frac{x}{x + 2} + \tfrac{1}{3}\left(\frac{x}{x + 2}\right)^3 + \tfrac{1}{5}\left(\frac{x}{x + 2}\right)^5 + \ldots\right\}. \qquad (\beta)$$

Write $\dfrac{1}{u}$ instead of x in (β), observing that

$$\log_\epsilon \left(1 + \frac{1}{u}\right) = \log_\epsilon \frac{u + 1}{u} = \log_\epsilon (u + 1) - \log_\epsilon u,$$

and we get

$$\log_\epsilon (u + 1) - \log_\epsilon u = 2 \left(\frac{1}{2u + 1} + \frac{1}{3(2u + 1)^3} + \frac{1}{5(2u + 1)^5} + \ldots\right).$$

27. Logarithms having ϵ for their base are called Napierian, from the name of Napier their inventor. Logarithms having any other base are deduced from the Napierian logarithms by means of the formula $\log_\epsilon a \,.\, \log_a x = \log_\epsilon x$, which may be investigated as follows:

$$\epsilon^{\log_\epsilon x} = x = a^{\log_a x} = (\epsilon^{\log_\epsilon a})^{\log_a x} = \epsilon^{\log_\epsilon a \,.\, \log_a x}; \quad \therefore \log_\epsilon x = \log_\epsilon a \,.\, \log_a x.$$

$(\log_\epsilon a)^{-1}$ is called the modulus of the system whose base is a.

The base of the system of logarithms commonly used is 10.

$$\log_\epsilon 10 = 2{,}302585093. \quad \log_{10}(\log_\epsilon 10) = 0.3622156887.$$

$$\log_\epsilon x = (\log_\epsilon 10). \quad \log_{10} x = (2{,}302585093)\log_{10} x.$$

$$\log_{10} x = (\log_\epsilon 10)^{-1}\log_\epsilon x = (0{,}4342944819)\log_\epsilon x.$$

28. $\left(\dfrac{\sin x}{x}\right)_{x=0} = 1.$

$\sin x > x$, and $\tan x > x$ as long as $x < \frac{1}{2}\pi$;

$$\therefore\ 1 = \frac{\sin x}{\sin x} > \frac{\sin x}{x}, \quad \cos x = \frac{\sin x}{\tan x} < \frac{\sin x}{x};$$

$$\therefore\ \frac{\sin x}{x} \text{ lies between } 1 \text{ and } \cos x.$$

But $(\cos x)_{x=0} = 1$, $\therefore\ \left(\dfrac{\sin x}{x}\right)_{x=0} = 1.$

29. 1. $d_x \sin x = \cos x.$

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$\frac{\sin(x+h) - \sin x}{h} = \cos x\,\frac{\sin h}{h} - \sin x\,\frac{1 - \cos h}{h}$$

$$= \cos x\,\frac{\sin h}{h} - \sin x\,\frac{2(\sin\frac{1}{2}h)^2}{h}$$

$$\left\{\frac{\sin(x+h) - \sin x}{h}\right\}_{h=0} = \cos x\left\{\frac{\sin h}{h}\right\}_{h=0} - \sin x\left\{\frac{2(\sin\frac{1}{2}h)^2}{h}\right\}_{h=0}$$

$$= \cos x.$$

For, $\left\{\dfrac{\sin h}{h}\right\}_{h=0} = 1$, $\left\{\dfrac{2(\sin\frac{1}{2}h)^2}{h}\right\}_{h=0} = \left\{\sin\tfrac{1}{2}h\,\dfrac{(\sin\frac{1}{2}h)}{\frac{1}{2}h}\right\}_{h=0} = 0$ (28).

And $d_x \sin x = \left\{\dfrac{\sin(x+h) - \sin x}{h}\right\}_{h=0} = \cos x.$

2. $d_x \cos x = -\sin x$.

$$(\cos x)^2 = 1 - (\sin x)^2,$$

$$\therefore\ 2\cos x\, d_x \cos_x = -2\sin x\, d_x \sin x = -2\sin x \cos x,$$

$$\therefore\ d_x \cos x = -\sin x.$$

3. $d_x \operatorname{ver\,sin} x = d_x(1 - \cos x) = \sin x$.

4. $d_x \tan x = (\operatorname{secant} x)^2$.

$$d_x \tan x = d_x \frac{\sin x}{\cos x} = \frac{1}{(\cos x)^2}\left\{\cos x\, d_x \sin x - \sin x\, d_x \cos x\right\}\ (15),$$

$$= \frac{1}{(\cos x)^2}\left\{(\cos x)^2 + (\sin x)^2\right\} = \frac{1}{(\cos x)^2} = (\operatorname{secant} x)^2.$$

5. $d_x \operatorname{cotan} x = -(\operatorname{cosec} x)^2$.

$$d_x \operatorname{cotan} x = d_x \frac{\cos x}{\sin x} = \frac{1}{(\sin x)^2}\left\{\sin x\, d_x \cos x - \cos x\, d_x \sin x\right\}$$

$$= -\frac{1}{(\sin x)^2}\left\{(\sin x)^2 + (\cos x)^2\right\} = -\frac{1}{(\sin x)^2} = -(\operatorname{cosec} x)^2.$$

6. $d_x \operatorname{secant} x = \tan x \operatorname{secant} x$.

$$d_x \operatorname{secant} x = d_x \frac{1}{\cos x} = -\frac{d_x \cos x}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} = \tan x \operatorname{secant} x.$$

7. $d_x \operatorname{cosec} x = -\operatorname{cotan} x \operatorname{cosec} x$.

$$d_x \operatorname{cosec} x = d_x \frac{1}{\sin x} = -\frac{d_x \sin x}{(\sin x)^2} = -\frac{\cos x}{(\sin x)^2} = -\operatorname{cotan} x \operatorname{cosec} x.$$

30. 1. $d_x \sin^{-1} x = \dfrac{1}{\sqrt{(1 - x^2)}}$.

Let $z = \sin^{-1} x$, $\therefore\ \sin z = x$, $\therefore\ \cos z\, d_x z = 1$,

and $\cos z = \sqrt{(1 - \overline{\sin z}|^2)} = \sqrt{(1 - x^2)}$;

$$\therefore\ \sqrt{(1 - x^2)}\, d_x \sin^{-1} x = 1;\ \therefore\ d_x \sin^{-1} x = \frac{1}{\sqrt{(1 - x^2)}}.$$

2. $d_x \cos^{-1} x = -\dfrac{1}{\sqrt{(1-x^2)}}$.

$$\text{Let } z = \cos^{-1} x, \ \therefore\ \cos z = x, \ \therefore\ -\sin z \, d_x z = 1,$$

$$\text{and } \sin z = \sqrt{(1 - \overline{\cos z}|^2)} = \sqrt{(1-x^2)};$$

$$\therefore\ -\sqrt{(1-x^2)}\, d_x \cos^{-1} x = 1; \ \therefore\ d_x \cos^{-1} x = -\frac{1}{\sqrt{(1-x^2)}}.$$

3. $d_x \operatorname{ver}\sin^{-1} x = \dfrac{1}{\sqrt{(2x-x^2)}}$.

$$\text{Let } z = \operatorname{ver}\sin^{-1} x, \ \therefore\ \operatorname{ver}\sin z = x, \ \therefore\ \sin z \, d_x z = 1,$$

$$\text{and } \sin z = \sqrt{(1 - \overline{\cos z}|^2)} = \sqrt{(1 - \overline{1 - \operatorname{ver}\sin z}|^2)}$$

$$= \sqrt{(1 - \overline{1-x}|^2)} = \sqrt{(2x - x^2)};$$

$$\therefore\ \sqrt{(2x-x^2)}\, d_x \operatorname{ver}\sin^{-1} x = 1; \ \therefore\ d_x \operatorname{ver}\sin^{-1} x = \frac{1}{\sqrt{2x - x^2}}.$$

4. $d_x \tan^{-1} x = \dfrac{1}{1+x^2}$.

$$\text{Let } z = \tan^{-1} x, \ \therefore\ \tan z = x, \ \therefore\ (\sec z)^2 \, d_x z = 1,$$

$$\text{and } (\sec z)^2 = 1 + (\tan z)^2 = 1 + x^2; \ \therefore\ (1+x^2)\, d_x \tan^{-1} x = 1;$$

$$\therefore\ d_x \tan^{-1} x = \frac{1}{1+x^2}.$$

5. $d_x \operatorname{cotan}^{-1} x = -\dfrac{1}{1+x^2}$.

$$\text{Let } z = \operatorname{cotan}^{-1} x, \ \therefore\ \operatorname{cotan} z = x, \ \therefore\ -(\operatorname{cosec} z)^2 \, d_x z = 1,$$

$$\text{and } (\operatorname{cosec} z)^2 = 1 + (\operatorname{cotan} z)^2 = 1 + x^2;$$

$$\therefore\ -(1+x^2)\, d_x \operatorname{cotan}^{-1} x = 1; \ \therefore\ d_x \operatorname{cotan}^{-1} x = -\frac{1}{1+x^2}.$$

6. $d_x \sec^{-1} x = \dfrac{1}{x\sqrt{(x^2-1)}}.$

Let $z = \sec^{-1} x$, $\therefore$ $\sec z = x$, $\therefore$ $\tan z \sec z \, d_x z = 1$, and

$$\tan z = \sqrt{(\overline{\sec z}|^2 - 1)} = \sqrt{(x^2 - 1)};$$

$$\therefore\ x\sqrt{(x^2 - 1)}\, d_x \sec^{-1} x = 1;$$

$$\therefore\ d_x \sec^{-1} x = \frac{1}{x\sqrt{(x^2 - 1)}}.$$

7. $d_x \operatorname{cosec}^{-1} x = -\dfrac{1}{x\sqrt{(x^2 - 1)}}.$

Let $z = \operatorname{cosec}^{-1} x$, $\therefore$ $\operatorname{cosec} z = x$,

$$\therefore\ -\operatorname{cotan} z \operatorname{cosec} z \, d_x z = 1,$$

and $\operatorname{cotan} z = \sqrt{(\overline{\operatorname{cosec} z}|^2 - 1)} = \sqrt{(z^2 - 1)};$

$$\therefore\ -x\sqrt{(x^2 - 1)}\, d_x \operatorname{cosec}^{-1} x = 1;$$

$$\therefore\ d_x \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{(x^2 - 1)}}.$$

31. If $u = \dfrac{x}{a}$, $d_x u = \dfrac{1}{a}$,

and $d_x f\left(\dfrac{x}{a}\right) = d_x f(u) = d_u f(u)\, d_x u = \dfrac{1}{a} d_u f u.$

Consequently

$$d_x \sin\frac{x}{a} = \frac{1}{a}\cos\frac{x}{a}.$$

$$d_x \cos\frac{x}{a} = -\frac{1}{a}\sin\frac{x}{a}.$$

$$d_x \operatorname{ver}\sin\frac{x}{a} = \frac{1}{a}\sin\frac{x}{a}.$$

$$d_x \tan \frac{x}{a} = \frac{1}{a}\left(\sec \frac{x}{a}\right)^2.$$

$$d_x \operatorname{cotan} \frac{x}{a} = -\frac{1}{a}\left(\operatorname{cosec} \frac{x}{a}\right)^2.$$

$$d_x \sec \frac{x}{a} = \frac{1}{a} \tan \frac{x}{a} \sec \frac{x}{a}.$$

$$d_x \operatorname{cosec} \frac{x}{a} = -\frac{1}{a} \operatorname{cotan} \frac{x}{a} \operatorname{cosec} \frac{x}{a}.$$

$$d_x \sin^{-1} \frac{x}{a} = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

$$d_x \cos^{-1} \frac{x}{a} = -\frac{1}{\sqrt{(a^2 - x^2)}}.$$

$$d_x \operatorname{ver} \sin^{-1} \frac{x}{a} = \frac{1}{\sqrt{(2ax - x^2)}}.$$

$$d_x \tan^{-1} \frac{x}{a} = \frac{a}{a^2 + x^2}.$$

$$d_x \operatorname{cotan}^{-1} \frac{x}{a} = -\frac{a}{a^2 + x^2}.$$

$$d_x \sec^{-1} \frac{x}{a} = \frac{a}{x\sqrt{(x^2 - a^2)}}.$$

$$d_x \operatorname{cosec}^{-1} \frac{x}{a} = -\frac{a}{x\sqrt{(x^2 - a^2)}}.$$

32. To find the successive differential coefficients of $\sin x$ and $\cos x$.

$$d_x \sin x = \cos x.$$

$$d_x^2 \sin x = -\sin x.$$

$$d_x^3 \sin x = -\cos x.$$

$$d_x^4 \sin x = \sin x.$$

$$\ldots\ldots = \ldots\ldots$$

$$d_x^{4n} \sin x = \sin x.$$

$$d_x^{4n+1} \sin x = \cos x.$$

$$d_x^{4n+2} \sin x = -\sin x.$$

$$d_x^{4n+3} \sin x = -\cos x.$$

$$d_x \cos x = -\sin x.$$

$$d_x^2 \cos x = -\cos x.$$

$$d_x^3 \cos x = \sin x.$$

$$d_x^4 \cos x = \cos x.$$

$$\ldots\ldots = \ldots\ldots$$

$$d_x^{4n} \cos x = \cos x.$$

$$d_x^{4n+1} \cos x = -\sin x.$$

$$d_x^{4n+2} \cos x = -\cos x.$$

$$d_x^{4n+3} \cos x = \sin x.$$

33. To expand $\cos x$, $\sin x$ in ascending powers of x.

Cos $(-x) = \cos x$, therefore the expansion of $\cos x$ contains none but even powers of x. Also $\{\cos x\}_{x=0} = 1$, therefore the first term of the series $= 1$. Consequently we may assume

$$\cos x = 1 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8 + \ldots$$

Differentiate, and we get

$$-\sin x = 2a_2x + 4a_4x^3 + 6a_6x^5 + 8a_8x_7 + \ldots$$

Differentiate again, and we get

$$-\cos x = 2a_2 + 4.3.a_4x^2 + 6.5.a_6x^4 + 8.7.a_8x^6 + \ldots$$

Therefore, equating the coefficients of like powers of x in the two expansions of $\cos x$,

$$a_2 = -\frac{1}{2} = -\frac{1}{1.2}, \quad a_4 = -\frac{a_2}{3.4} = \frac{1}{1.2.3.4},$$

$$a_6 = -\frac{a_4}{5.6} = -\frac{1}{1.2.3.4.5.6}, \text{ \&c.} = \text{\&c.}$$

$$\therefore \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \ldots$$

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \ldots$$

$$\epsilon^{x\sqrt{-1}} = 1 + x\sqrt{-1} - \frac{x^2}{1.2} - \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \frac{x^6}{1.2.3.4.5.6} - \ldots$$

$$\epsilon^{-x\sqrt{-1}} = 1 - x\sqrt{-1} - \frac{x^2}{1.2} + \frac{x^3\sqrt{-1}}{1.2.3} + \frac{x^4}{1.2.3.4} - \frac{x^5\sqrt{-1}}{1.2.3.4.5} - \frac{x^6}{1.2.3.4.5.6} - \ldots$$

$$\therefore \tfrac{1}{2}\left\{\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}\right\} = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \ldots = \cos x.$$

$$\frac{1}{2\sqrt{-1}}\left\{\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}\right\} = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots = \sin x.$$

i. e. the expansion of $\frac{1}{2}\left\{\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}\right\}$ is identical with that of $\cos x$, and the expansion of $\frac{1}{2\sqrt{-1}}\left\{\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}\right\}$ with that of $\sin x$.

$$\tan x = \frac{\sin x}{\cos x} = \frac{1}{\sqrt{-1}}\,\frac{\epsilon^{x\sqrt{-1}} - \epsilon^{-x\sqrt{-1}}}{\epsilon^{x\sqrt{-1}} + \epsilon^{-x\sqrt{-1}}} = \frac{1}{\sqrt{-1}}\,\frac{\epsilon^{2x\sqrt{-1}} - 1}{\epsilon^{2x\sqrt{-1}} + 1}.$$

34. To expand $\sin^{-1} x$ in a series of ascending powers of x, $\sin^{-1} x$ being less than a right angle.

Since $\{\sin^{-1} x\}_{x=0} = 0$, every term of the expansion of $\sin^{-1} x$ is multiplied by a positive power of x. We may therefore assume

$$\sin^{-1} x = Ax^a + Bx^b + Cx^c + Dx^d + \dots$$

therefore, differentiating with respect to x,

$$\frac{1}{\sqrt{(1-x^2)}} = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} + dDx^{d-1} + \dots$$

But $\dfrac{1}{\sqrt{(1-x^2)}} = 1 + \dfrac{1}{2}x^2 + \dfrac{1.3}{2.4}x^4 + \dfrac{1.3.5}{2.4.6}x^6 + \dots$

therefore, equating the terms taken in order,

$$a-1=0,\ A=1;\ b-1=2,\ B=\frac{1}{2}\frac{1}{3};\ c-1=4,\ C=\frac{1.3}{2.4}\frac{1}{5};$$

$$d-1=6,\ D=\frac{1.3.5}{2.4.6}\frac{1}{7};\ \dots$$

$$\therefore\ \sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \dots$$

35. To expand $\tan^{-1} x$ in a series of ascending powers of x, $\tan^{-1} x$ being less than a right angle.

Since $\{\tan^{-1} x\}_{x=0} = 0$, every term of the expansion of $\tan^{-1} x$ is multiplied by a positive power of x. We may therefore assume

$$\tan^{-1} x = Ax^a + Bx^b + Cx^c + Dx^d + \dots$$

therefore, differentiating with respect to x,

$$\frac{1}{1+x^2} = aAx^{a-1} + bBx^{b-1} + cCx^{c-1} + dDx^{d-1} + \dots$$

$$\text{But } \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

therefore, equating the terms taken in order,

$a - 1 = 0$, $A = 1$; $b - 1 = 2$, $B = -\frac{1}{3}$; $c - 1 = 4$, $C = \frac{1}{5}$;

$d - 1 = 6$, $D = -\frac{1}{7}$; ...

$$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

26. The ratio of the circumference of a circle to its diameter has been computed by means of the expression

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

For, since the tangent of half a right angle = 1,

$$\frac{\pi}{4} = (\tfrac{1}{8} \text{ circumference}) \div \text{radius} = \tan^{-1} 1.$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = 2\left(\frac{1}{1\,.\,3} + \frac{1}{5\,.\,7} + \dots\right).$$

This series converges too slowly to be employed conveniently in calculating the value of π. Euler transformed it into a rapidly converging series by means of the equation $\tan^{-1} 1 = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$, which gives

$$\frac{\pi}{4} = \left\{ \begin{array}{l} \frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \frac{1}{7\,.\,2^7} + \dots \\ + \frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \frac{1}{7\,.\,3^7} + \dots \end{array} \right\}.$$

And Machin transformed it into one still more highly convergent in the following manner:

$$\tan^{-1} 1 = 4 \tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239};$$

$$\therefore \frac{\pi}{4} = \left\{ \begin{array}{c} 4\left(\frac{1}{5} - \frac{1}{3\,.\,5^3} + \frac{1}{5\,.\,5^5} - \frac{1}{7\,.\,5^7} + \dots\right. \\ -\left(\frac{1}{239} - \frac{1}{3\,.\,(239)^3} + \frac{1}{5\,.\,(239)^5} - \frac{1}{7\,.\,(239)^7} + \dots\right. \end{array} \right\}.$$

In this way it is found that

$$\pi = 3{,}14159265358979 3, \ \&c.$$

SECTION II.

EXPANSION.

37. $d_x f(x+h) = d_h f(x+h)$:

where $f(x+h)$ denotes any function of $x+h$, and x and h are independent of each other.

Let $z = x + h$, $\therefore\ d_x z = 1$, $d_h z = 1$.

$$d_x f(z) = d_z f(z)\, d_x z = d_z f(z).$$

$$d_h f(z) = d_z f(z)\, d_h z = d_z f(z).$$

$$\therefore\ d_x f(z) = d_h f(z) \quad \text{or}$$

$$d_x f(x+h) = d_h f(x+h).$$

So also

$$d_x^2 f(x+h) = d_h^2 f(x+h).$$

$$d_x^3 f(x+h) = d_h^3 f(x+h).$$

$$\dots\dots\dots\dots = \dots\dots\dots\dots$$

$$d_x^n f(x+h) = d_h^n f(x+h).$$

38. Taylor's Theorem. To expand $f(x+h)$ in a series of ascending powers of h; $f(x+h)$ being the same function of $x+h$, that $f(x)$ is of x.

It appears from (2) that we may assume

$$f(x+h) = f(x) + Ah + Bh^b + Ch^c + Dh^d + \dots$$

where A, B, C, ... do not contain h, and b, c, ... are arranged in ascending order.

$$d_h f(x+h) = A + bBh^{b-1} + cCh^{c-1} + dDh^{d-1} + \dots$$
$$d_x f(x+h) = d_x f(x) + d_x Ah + d_x Bh^b + d_x Ch^c + \dots$$

But $d_h f(x+h) = d_x f(x+h)$ (37); therefore, equating the terms taken in order,

$$A = d_x f(x).$$

$$bBh^{b-1} = d_x Ah, \ \therefore b = 2, \ B = \tfrac{1}{2} d_x A = \frac{1}{1\,.\,2} d_x^2 f(x).$$

$$cCh^{c-1} = d_x Bh^b, \ \therefore c = 3, \ C = \tfrac{1}{3} d_x B = \frac{1}{1\,.\,2\,.\,3} d_x^3 f(x).$$

$$dDh^{d-1} = d_x Ch^c, \ \therefore d = 4, \ D = \tfrac{1}{4} d_x C = \frac{1}{1\,.\,2\,.\,3\,.\,4} d_x^4 f(x).$$

……

$$\therefore f(x+h) = f(x) + d_x f(x)\frac{h}{1} + d_x^2 f(x)\frac{h^2}{1.2} + d_x^3 f(x)\frac{h^3}{1.2.3} + \dots$$

This series may be exhibited in an abbreviated form analagous to the exponential expressions for sin x and cos x in (33). For, if we expand ϵ^{hd_x} by (20), treating d_x as a symbol of quantity, the powers of h and their divisors are identical in the two series, and the powers of d_x in one series are the same as the number of differentiations in the other. Hence the theorem may be written in the following manner

$$f(x+h) = \epsilon^{hd_x} f(x).$$

39. Maclaurin's Theorem. To expand $f(x)$ in a series of ascending powers of x.

In (38) let $x = 0$, denoting by $f_{x=0}(x)$, $d_{x=0}f(x)$, $d_{x=0}^2 f(x)$,.. the values of $f(x)$, $d_x f(x)$, $d_x^2 f(x)$,... when $h = 0$, and we get

$$f(h) = f_{x=0}(x) + d_{x=0}f(x)\,h + d_{x=0}^2 f(x)\frac{h^2}{1\,.\,2} + d_{x=0}^3 f(x)\frac{h^3}{1\,.\,2\,.\,3} +$$

therefore, writing x instead of h,

$$f(x) = f_{x=0}(x) + d_{x=0}f(x)\,x + d_{x=0}^2 f(x)\frac{x^2}{1\,.\,2} + d_{x=0}^3 f(x)\frac{x^3}{1\,.\,2\,.\,3} +$$

This theorem may also be deduced in the following manner; taking for granted the possibility of expanding $f(x)$ in a series of positive integral powers of x.

$$\begin{aligned} \text{Let } f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \\ \therefore\ d_x f(x) &= a_1 + 2a_2 x^2 + 3a_3 x^2 + \ldots \\ d_x^2 f(x) &= 2a_2 + 3.2.a_3 x + \ldots \\ d_x^3 f(x) &= 3.2.a_3 + \ldots \\ \ldots\ldots\ldots &= \ldots \end{aligned}$$

Make $x = 0$, and we get

$$f_{x=0}(x) = a_0,\ d_{x=0} f(x) = a_1,\ d_{x=0}^2 f(x) = 2a_2,\ d_{x=0}^3 f(x) = 3.2.a_3, \ldots$$

$$\therefore\ f(x) = f_{x=0}(x) + d_{x=0} f(x)\, x + d_{x=0}^2 f(x) \frac{x^2}{1.2} + d_{x=0}^3 f(x) \frac{x^3}{1.2.3} + \ldots$$

40. Cases in which Taylor's Theorem fails.

If $f(x)$ or any of its differential coefficients become infinite when x takes a particular value a, it is manifest that the true expansion of $f(x+h)$, when $x = a$, cannot be obtained by Taylor's Theorem.

As instances we may take $\left\{\sec(x+h)\right\}_{x=\frac{\pi}{2}}$; for $\sec\frac{\pi}{2} = \infty$:

and $\left\{(x+h-a)^{n+\frac{r}{s}}\right\}_{x=a}$, for $d_x^{n+1}(x-a)^{n+\frac{r}{s}}$ involves $(x-a)^{-\left(1-\frac{r}{s}\right)}$, which, r being less than s, becomes infinite when $x = a$.

In these cases the true expansion of $f(a+h)$ is found to contain negative or fractional powers of h. Thus, secant $(\frac{1}{2}\pi+h)$ $= -\operatorname{cosec} h = -(\sin h)^{-1} = -\left(h - \frac{h^3}{1.2.3} + ..\right)^{-1} = -h^{-1} - \frac{h}{1.2.3} + ..$

And $(a + h - a)^{n+\frac{r}{s}} = h^{n+\frac{r}{s}}$.

In order to prove generally that some of the quantities $f_{x=a}(x)$, $d_{x=a}f(x)$ ... are infinite, or that Taylor's theorem fails, when the true expansion of $f(a+h)$ involves negative or fractional powers of h, let

$$f(a+h) = \dots\dots + Nh^n + \dots\dots Sh^\sigma + \dots\dots$$

where ... n, ... σ, ... are arranged in ascending order, and σ, the lowest fractional exponent, lies between $s-1$ and s.

$$d_x^n f(x+h) = d_h^n f(x+h) \ (37), \ \therefore \ d_{x=a}^n f(x+h) = d_h^n f(a+h).$$

and $f_{x=a}(x+h) = \dots + Nh^n + \dots + Sh^\sigma + \dots$

$$\therefore \ d_{x=a} f(x+h) = \dots + nNh^{n-1} + \dots + \sigma S h^{\sigma-1} + \dots$$

$$\dots\dots\dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$d_{x=a}^n f(x+h) = n(n-1).3.2.1N + . + \sigma(\sigma-1).(\sigma-n+1)Sh^{\sigma-n} + .$$

$$\dots\dots\dots\dots\dots = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$d_{x=a}^s f(x+h) = \qquad\qquad + \sigma(\sigma-1)..(\sigma-s+1)Sh^{\sigma-s} + ..$$

From which we obtain the values of $f_{x=a}(x)$, $d_{x=a}f(x)$,... $d_{x=a}^n f(x)$,... by making $h=0$.

All the terms that follow N in the expression for $d_{x=a}^n f(x+h)$ vanish when h vanishes,

$$\therefore \ d_{x=0}^n f(x) = n(n-1) \dots 3.2.1.N.$$

Hence, the coefficient of h^n, and therefore that of every power of h, less than h^σ, is correctly determined by the Theorem.

Also, since $\sigma - s$ is negative, $h^{\sigma-s}$ becomes infinite when $h=0$,

$$\therefore \ d_{x=a}^s f(x) = \infty.$$

And all the following differential cofficients contain negative powers of h, and are therefore infinite when $h=0$.

When the expansion of $f(a+h)$ contains a negative power of h, $d_h f(a+h)$, ... $d_h^n f(a+h)$, ... must each contain a negative power of h, which becomes infinite when $h=0$, and therefore $f_{x=a}(x)$, $d_{x=a} f(x)$, ... $d_{x=a}^n f(x)$, ... are all infinite.

Hence also, if $d_{x=a}^s f(x)$ is the first differential coefficient that becomes infinite, the true expansion of $f(x+h)_{x=a}$ contains a fractional power of h, the exponent of which lies between $s-1$ and s. And, if $f(x)_{x=a}=\infty$, the true expansion of $f(x+h)_{x=a}$ contains a negative power of h.

When any of the quantities $f(x)_{x=a}$, $d_{x=a} f(x)$, ... become infinite we must expand $f(a+h)$ by the common algebraic methods.

Maclaurin's series is deduced from Taylor's by making $x=0$ in the expansion of $f(x+h)$, and therefore fails when any of the quantities $f(x)_{x=0}$, $d_{x=0} f(x)$, ... become infinite.

41. The following investigation of Taylor's Theorem possesses the advantage of exhibiting the difference between $f(x+h)$ and the first $n+1$ terms of its expansion.

Let $f(x)$ be any function of x, and $f(z)$ the same function of z; and let $P=\{f(z)-f(x)\}\div(z-x)$, therefore

$$f(z)=f(x)+P(z-x).$$

$$0=d_x f(x)+d_x P(z-x)-P. \qquad (1)$$

$$0=d_x^2 f(x)+d_x^2 P(z-x)-2d_x P. \qquad (2)$$

$$0=d_x^3 f(x)+d_x^3 P(z-x)-3d_x^2 P. \qquad (3)$$

$$\cdots = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0=d_x^n f(x)+d_x^n P(z-x)-n d_x^{n-1} P. \qquad (n)$$

Multiplying (1), (2), (3), ... (n) by $\frac{(z-x)}{\lfloor\underline{1}}$, $\frac{(z-x)^2}{\lfloor\underline{2}}$, $\frac{(z-x)^3}{\lfloor\underline{3}}$, ... $\frac{(z-x)^n}{\lfloor\underline{n}}$, respectively, where $\lfloor\underline{1}$, $\lfloor\underline{2}$, $\lfloor\underline{3}$, .. $\lfloor\underline{n}$, denote the products of the natural numbers, 1, 1.2, 1.2.3,... 1.2.3...*n*, respectively, and adding, we obtain

$$f(z)=f(x)+d_x f(x)\frac{(z-x)}{\underline{|1}}+d_x^2 f(x)\frac{(z-x)^2}{\underline{|2}}+d_x^3 f(x)\frac{(z-x)^3}{\underline{|3}}+\ldots$$

$$\ldots\ldots+d_x^n f(x)\frac{(z-x)^n}{\underline{|n}}+d_x^n P\frac{(z-x)^n}{\underline{|n}}.$$

If $z-x=h$, this becomes

$$f(x+h)=f(x)+d_x f(x)\frac{h}{\underline{|1}}+d_x^2 f(x)\frac{h^2}{\underline{|2}}+d_x^3 f(x)\frac{h^3}{\underline{|3}}+\ldots$$

$$+d_x^n f(x)\frac{h^n}{\underline{|n}}+d_x^n P\frac{h^n}{\underline{|n}}.$$

42. In the series $a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$ any term $a_n n^n$ may be made greater than the sum of all that follow by diminishing x.

The most unfavourable case is when all the terms are positive, their number infinite, and each coefficient equal to the greatest. In this case, if the greatest coefficient $=k$, the sum of all the terms that follow $=x^n kx\div(1-x)$; and by diminishing x, $kx\div(1-x)$ may be made as small as we please, and therefore less than a_n. Therefore a_nx^n may be made greater than $x^n kx\div(1-x)$, and, consequently, greater than the sum of all the terms that follow in every case.

When one of the terms following a_nx^n contains a power of x, the exponent of which is the fraction $\frac{r}{s}$, write z^s instead of x. The exponents of z are integers, therefore by the former case any term may be made greater than the sum of all that follow by diminishing z.

43. $\{f(b).-f(a)\}\div(b-a)$ and $d_x f(x)$ have the same sign as long as x lies between a and b, provided the sign of $d_x f(x)$ remains unaltered while x changes continuously from a to b.

Let $b - a = nh$, and

$$f(a+h) - f(a) = P_1 h$$
$$f(a+2h) - f(a+h) = P_2 h$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f(b) - f\{a + (n-1)h\} = P_n h;$$
$$\therefore f(b) - f(a) = (P_1 + P_2 + \ldots + P_n)h;$$
$$\therefore \{f(b) - f(a)\} \div (b-a) = \frac{1}{n}(P_1 + P_2 + \ldots + P_n).$$

Hence, if $P_1, P_2, \ldots P_n$, have all the same sign,

$$\{f(b) - f(a)\} \div (b-a)$$

will be positive or negative according as $P_1, P_2, \ldots P_n$ are positive or negative.

Now $P_1 = f'(a) + f''(a) h^{\beta-1} + \ldots$ (2), and by diminishing h the first term of this series may be made greater than the sum of all that follow (42), $\therefore$ P_1 may be made to have the same sign as $f'(a)$ or $d_{x=a} f(x)$. In the same manner each of the quantities $P_2, \ldots P_n$ may be made to have the same sign as the corresponding value of $d_x f(x)$. Consequently, when the sign of $d_x f(x)$ remains unaltered while x changes continuously from a to b, if we take n sufficiently large, $P_1, P_2, \ldots P_n$ will have each the same sign as $d_x f(x)$, and $\therefore$ $\{f(b) - f(a)\} \div (b-a)$ will be positive or negative according as $d_x f(x)$ is positive or negative between the prescribed limits.

44. To find the limits of the difference between $f(x+h)$ and the first $n+1$ terms of its expansion by Taylor's series.

Retaining the notation of (41), and writing $n+1$ instead of n, we have

$$0 = d_x^{n+1} f(x) + d_x^{n+1} P(z-x) - (n+1) d_x^n P;$$
$$\therefore 0 = d_x^{n+1} f(x)(z-x)^n + d_x^{n+1} P(z-x)^{n+1} - (n+1) d_x^n P(z-x)^n$$
$$= d_x^{n+1} f(x)(z-x)^n + d_x\{d_x^n P(z-x)^{n+1}\}.$$

Let A, B be the greatest and least values which $d_x^{n+1}f(x)$ can take while x changes from x_1 to z; then, since

$$0 = d_x^{n+1}f(x)\,(z-x)^n + d_x\left\{d_x^n P\,(z-x)^{n+1}\right\},$$

one of the quantities

$$A(z-x)^n + d_x\left\{d_x^n P(z-x)^{n+1}\right\},$$

$$B(z-x)^n + d_x\left\{d_x^n P(z-x)^{n+1}\right\}$$

can never be positive, and the other can never be negative, as long as x lies between x_1 and z.

$$\text{If } F(x) = -A\frac{(z-x)^{n+1}}{n+1} + d_x^n P(z-x)^{n+1},$$

$$F(z) = 0;$$

$$\therefore \left\{F(z) - F(x)\right\} \div (z-x) = A\frac{(z-x)^n}{n+1} - d_x^n P(z-x)^n,$$

$$\text{and } d_x F(x) = A(z-x)^n + d_x\left\{d_x^n P(z-x)^n\right\}.$$

Now, as long as x lies between x_1 and z, the sign of $d_x F(x)$ will not change, provided the sign of $d_x^{n+1}f(x)$ be invariable; therefore, as long as x lies between x_1 and z, $\left\{F(z) - F(x_1)\right\} \div (z-x_1)$ and $d_x F(x)$ have the same sign (43); therefore

$$A\frac{(z-x_1)^n}{n+1} - d_{x=x_1}^n P(z-x_1)^n \text{ and } A(z-x)^n + d_x\left\{d_x^n P(z-x)^{n+1}\right\}$$

have the same sign. In like manner

$$B\frac{(z-x_1)^n}{n+1} - d_{x=x_1}^n P(z-x_1)^n \text{ and } B(z-x)^n + d_x\left\{d_x^n P(z-x)^{n+1}\right\}$$

have the same sign. Therefore, one of the quantities

$$\frac{A}{n+1} - d_{x=x_1}^n P,\quad \frac{B}{n+1} - d_{x=x_1}^n P,$$

can never be positive, and the other can never be negative; or $\frac{A}{n+1}$, $\frac{B}{n+1}$ are the limits of the value of $d_{x=x_1}^n P$.

Hence, if $x - x_1 = h$, $d^n_{x=x_1} P \frac{h^{n+1}}{\lfloor\underline{n}}$, the remainder after the first $n+1$ terms of the expansion of $f(x_1+h)$ (41), lies between $A\frac{h^{n+1}}{\lfloor\underline{n+1}}$ and $B\frac{h^{n+1}}{\lfloor\underline{n+1}}$, or, between the greatest and least values which the first of the neglected terms can assume while x changes from x_1 to x_1+h, provided that term do not change its sign at the same time.

45. $\frac{1}{\lfloor\underline{n}} d^n_x f(x)$ is the coefficient of h^n in the expansion of $f(x+h)$ (41); if, therefore, we can by any means expand $f(x+h)$ in a series of the form $u_0 + u_1 h + \ldots + u_n h^n + \ldots$, we know that $d^n_x f(x) = \lfloor\underline{n} \,.\, u_n$, and thus obtain the n^{th} differential coefficient of $f(x)$ at once, instead of deducing it from $f(x)$ by n successive differentiations.

46. Application of Taylor's theorem to functions of two independent variables.

To expand $f(x+h, y+k)$ in a series of ascending powers of h and k; $f(x, y)$ being any function of the independent quantities x and y, and $f(x+h, y+k)$ the same function of $x+h$ and $y+k$.

$$f(x,y+k)=f(x,y)+d_y f(x,y)\frac{k}{1}+d_y^2 f(x,y)\frac{k^2}{\lfloor\!\underline{2}}+d_y^3 f(x,y)\frac{k^3}{\lfloor\!\underline{3}}+\ldots+d_y^r f(x,y)\frac{k^r}{\lfloor\!\underline{r}}+\ldots$$

Write $x+h$ instead of x, and we obtain

$$f(x+h,y+k)=f(x+h,y)+d_y f(x+h,y)\frac{k}{1}+d_y^2 f(x+h,y)\frac{k^2}{\lfloor\!\underline{2}}+d_y^3 f(x+h,y)\frac{k^3}{\lfloor\!\underline{3}}+\ldots+d_y^r f(x+h,y)\frac{k^r}{\lfloor\!\underline{r}}+\ldots$$

$$f(x+h,y)=f(x,y)+d_x f(x,y)\frac{h}{1}+d_x^2 f(x,y)\frac{h^2}{\lfloor\!\underline{2}}+d_x^3 f(x,y)\frac{k^3}{\lfloor\!\underline{3}}+\ldots+d_x^n f(x,y)\frac{h^n}{\lfloor\!\underline{n}}+\ldots$$

$$d_y f(x+h,y)=\quad d_y f(x,y)+d_x d_y f(x,y)\frac{h}{1}+d_x^2 d_y f(x,y)\frac{h^2}{\lfloor\!\underline{2}}+\ldots+d_x^{n-1} d_y f(x,y)\frac{h^{n-1}}{\lfloor\!\underline{n-1}}+\ldots$$

$$d_y^2 f(x+h,y)=\qquad d_y^2 f(x,y)+d_x d_y^2 f(x,y)\frac{h}{1}+\ldots+d_x^{n-2} d_y^2 f(x,y)\frac{h^{n-2}}{\lfloor\!\underline{n-2}}+\ldots$$

$$d_y^3 f(x+h,y)=\qquad d_y^3 f(x,y)+\ldots+d_x^{n-3} d_y^3 f(x,y)\frac{h^{n-3}}{\lfloor\!\underline{n-3}}+\ldots$$

$$\ldots\ldots\ldots\ldots\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$d_y^r f(x+h,y)=\qquad \ldots+d_x^{n-r} d_y^r f(x,y)\frac{h^{n-r}}{\lfloor\!\underline{n-r}}+\ldots$$

$$\ldots\ldots\ldots\ldots\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$
\begin{aligned}
\therefore\ f(x+h, y+k) = f(x, y) &+ d_x f(x, y)\frac{h}{1} + d_x^2 f(x, y)\frac{h^2}{\lfloor\underline{2}} + d_x^3 f(x, y)\frac{h^3}{\lfloor\underline{3}} + \ldots\ldots\ldots\ldots + d_x^n f(x, y)\frac{h^n}{\lfloor\underline{n}} + \ldots \\
&+ d_y f(x, y)\frac{k}{1} + d_x d_y f(x, y)\frac{h}{1}\,\frac{k}{1} + d_x^2 d_y f(x, y)\frac{h^2}{\lfloor\underline{2}}\,\frac{k}{1} + \ldots\ldots + d_x^{n-1} d_y f(x, y)\frac{h^{n-1}}{\lfloor\underline{n-1}}\,\frac{k}{1} + \ldots \\
&+ d_y^2 f(x, y)\frac{k^2}{\lfloor\underline{2}} + d_x d_y^2 f(x, y)\frac{h}{1}\,\frac{k^2}{\lfloor\underline{2}} + \ldots\ldots + d_x^{n-2} d_y^2 f(x, y)\frac{h^{n-2}}{\lfloor\underline{n-2}}\,\frac{k^2}{\lfloor\underline{2}} + \ldots \\
&+ d_y^3 f(x, y)\frac{k^3}{\lfloor\underline{3}} + \ldots\ldots\ldots + d_x^{n-3} d_y^3 f(x, y)\frac{h^{n-3}}{\lfloor\underline{n-3}}\,\frac{k^3}{\lfloor\underline{3}} + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots + d_x^{n-r} d_y^r f(x, y)\frac{h^{n-r}}{\lfloor\underline{n-r}}\,\frac{k^r}{\lfloor\underline{r}} + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

47. If we had commenced by writing down the series

$$
f(x+h, y) = f(x, y) + d_x f(x, y)\,h + d_x^2 f(x, y)\frac{h^2}{\lfloor\underline{2}} + d_x^3 f(x, h)\frac{h^3}{\lfloor\underline{3}} + \ldots + d_x^{n-r} f(x, y)\frac{h^{n-r}}{\lfloor\underline{n-r}},
$$

and had then substituted $y+k$ for y, we should have obtained

$$
\begin{aligned}
f(x+h, y+k) = f(x,y) &+ d_y f(x,y)\frac{k}{1} + d_y^2 f(x,y)\frac{k^2}{\underline{|2}} + d_y^3 f(x,y)\frac{k^3}{\underline{|3}} + \dots\dots\dots + d_y^n f(x,y)\frac{k^n}{\underline{|n}} + \dots \\
&+ d_x f(x,y)\frac{h}{1} + d_y d_x f(x,y)\frac{k}{1}\,\frac{h}{1} + d_y^2 d_x f(x,y)\frac{k^2}{\underline{|2}}\,\frac{h}{1} + \dots + d_y^{n-1} d_x f(x,y)\frac{k^{n-1}}{\underline{|n-1}}\,\frac{h}{1} + \dots \\
&+ d_x^2 f(x,y)\frac{h^2}{\underline{|2}} + d_y d_x^2 f(x,y)\frac{k}{1}\,\frac{h^2}{\underline{|2}} + \dots + d_y^{n-2} d_x^2 f(x,y)\frac{k^{n-2}}{\underline{|n-2}}\,\frac{h^2}{\underline{|2}} + \dots \\
&+ d_x^3 f(x,y)\frac{h^3}{\underline{|3}} + \dots\dots + d_y^{n-3} d_x^3 f(x,y)\frac{k^{n-3}}{\underline{|n-1}}\,\frac{h^3}{\underline{|3}} + \dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots + d_y^r d_x^{n-r} f(x,y)\frac{k^r}{\underline{|r}}\,\frac{h^{n-r}}{\underline{|n-r}} + \dots \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

These two expansions of $f(x+h, y+k)$ must be identical; therefore, equating the coefficients of like powers of h and k, we obtain

$$d_x d_y f(x,y) = d_y d_x f(x,y), \quad d_x^2 d_y f(x,y) = d_y d_x^2 f(x,y), \dots\dots \; d_x^{n-r} d_y^r f(x,y) = d_y^r d_x^{n-r} f(x,y).$$

Hence, if we differentiate $f(x, y)$ m times with respect to x, and n times with respect to y, the result will always be the same, whatever be the order in which the differentiations are performed.

It is easily seen that the $(n-1)^{\text{th}}$ column of the preceding expansion is represented by

$$\frac{(hd_x + kd_y)^n}{\underline{|n}} f(x, y),$$

and that, consequently, the theorem is represented as in (38) by the expression

$$f(x+h, y+k) = \epsilon^{hd_x + kd_y} f(x, y).$$

Cor. 1. Since $d_x^m d_y^n f(x, y)$ is the coefficient of $\frac{h}{\underline{|m}}\,\frac{k}{\underline{|n}}$ in the expansion of $f(x+h, y+k)$, it follows as in (45) that if we can expand $f(x+h, y+k)$ in a series of the form

$$u_0 + u_{1,0}h + u_{0,1}k + \dots + u_{m,n}h^m k^n + \dots,\ d_x^m d_y^n f(x, y) = \underline{|m}\,\underline{|n}\,.\,u_{m,n}.$$

Cor. 2. Exactly in the same manner it might be proved that the expansion of $f(x_1 + h_1, x_2 + h_2, \dots x_n + h_n)$, where $x_1, x_2, \dots x_n$ are independent quantities, is represented by the expression

$$f(x_1 + h_1, x_2 + h_2, \dots x_n + h_n) = \epsilon^{h_1 d_{x_1} + h_2 d_{x_2} + \dots + h_n d_{x_n}} . f(x_1, x_2, \dots x_n).$$

48. Application of Maclaurin's theorem to functions of two variables.

If in the expansion of $f(x+h, y+k)$ we make $x=0$, $y=0$, and afterwards write x instead of h, and y instead of k, we obtain

$$f(x,y) = f_{x=0,y=0}(x,y) + d_{x=0}f_{y=0}(x,y)\frac{x}{\underline{\lfloor 1}} + d^2_{x=0}f_{y=0}(x,y)\frac{x^2}{\underline{\lfloor 2}} + \dots\dots\dots + d^n_{x=0}f_{y=0}(x,y)\frac{x^n}{\underline{\lfloor n}} + \dots$$

$$+ d_{y=0}f_{x=0}(x,y)\frac{y}{\underline{\lfloor 1}} + d_{x=0}d_{y=0}f(x,y)\frac{x}{\underline{\lfloor 1}}\,\frac{y}{\underline{\lfloor 1}} + \dots + d^{n-1}_{x=0}d_{y=0}f(x,y)\frac{x^{n-1}}{\underline{\lfloor n-1}}\,\frac{y}{\underline{\lfloor 1}} + \dots$$

$$+ d^2_{y=0}f_{x=0}(x,y)\frac{y^2}{\underline{\lfloor 2}} + \dots\dots + d^{n-2}_{x=0}d^2_{y=0}f(x,y)\frac{x^{n-2}}{\underline{\lfloor n-1}}\,\frac{y^2}{\underline{\lfloor 2}} + \dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots + d^{n-r}_{x=0}d^r_{y=0}f(x,y)\frac{x^{n-r}}{\underline{\lfloor n-r}}\,\frac{y^r}{\underline{\lfloor r}} + \dots$$

$$\dots\dots\dots\dots\dots\dots$$

The same result might have been obtained by assuming, as in (39),

$$f(x,y) = a_0 + a_1x + b_1y + a_2x^2 + b_2xy + c_2y^2 + \dots$$

$$\therefore\ d_xf(x,y) = a_1 + 2a_2x + b_2y + \dots$$

$$d_yf(x,y) = b_1 + b_2x + 2c_2y + \dots$$

$$d_x^2 f(x,y) = \qquad 2a_2 + \ldots$$

$$d_x d_y f(x,y) = \qquad b_2 + \ldots$$

$$d_y^2 f(x,y) = \qquad 2c_2 + \ldots$$

therefore, making $x = 0$, $y = 0$,

$$f_{x=0,y=0}(x,y) = a_0,\ d_{x=0} f_{y=0}(x,y) = a_1,\ d_{y=0} f_{x=0}(x,y) = b_1,$$

$$d^2_{x=0} f_{y=0}(x,y) = 2a_2,\ d_{x=0} d_{y=0} f(x,y) = b_2,\ d^2_{y=0} f_{x=0}(x,y) = 2c_2, \ldots$$

49. $d_x\{F(v)d_z v\} = d_z\{F(v)d_x v\}$; where v is any function of the independent quantities x and z, and $F(v)$ any function of v.

Let w be such a function of v that $d_v w = F(v)$; then

$$d_x\{d_v w d_z v\} = d_x d_z w = d_z d_x w = d_z\{d_v w d_x v\};$$

$$\therefore\ d_x\{F(v)d_z v\} = d_z\{F(v)d_x v\}.$$

50. Lagrange's Theorem. To expand $u = f(y)$, where $y = z + x\phi(y)$, and x and z are independent of each other, in a series of ascending powers of x.

$$y = z + x\phi(y);$$

$$\therefore\ d_x y = \phi(y) + x d_y \phi(y) d_x y,\ d_z y = 1 + x d_y \phi(y) d_z y;$$

$$\therefore\ d_x y\{1 - x d_y \phi(y)\} = \phi(y),\ d_z y\{1 - x d_y \phi(y)\} = 1;$$

$$\therefore\ d_x y = \phi(y) d_z y.$$

$$d_x u = d_y u d_x y = \phi(y) d_y u d_z y = \phi(y) d_z u.$$

$$d_x^2 u = d_x\{\phi(y) d_z u\} = d_z\{\phi(y) d_x u\}(49) = d_z\{\overline{\phi(y)}|^2 d_z u\}$$

$$d_x^3 u = d_z d_x\{\overline{\phi(y)}|^2 d_z u\} = d_z d_z\{\overline{\phi(y)}|^2 d_x u\} = d_z^2\{\overline{\phi(y)}|^3 d_z u\}.$$

……

If $d_x^n u = d_z^{n-1}\{\overline{\phi(y)}|^n d_z u\}$

$$d_x^{n+1}u = d_z^{n-1}d_x\{\overline{\phi(y)}|^n d_z u\} = d_z^{n-1}d_z\{\overline{\phi(y)}|^n d_x u\} = d_z^n\{\overline{\phi(y)}|^{n+1} d_z u\};$$

Therefore, if the assumed value of $d_x u$ be true for any value of n, it is also true for the next higher value of n. But it is true for $d_x u$, $d_x^2 u$, $d_x^3 u$, therefore by successive inferences it is true for $d_x^n u$.

If we make $x=0$ in the equation $y = z + x\phi(y)$, y, $\phi(y)$, $d_z u$ become z, $\phi(z)$, $d_z f(z)$, respectively; therefore

$$u_{x=0} = f(z),\ d_{x=0}u = \phi(z)\,d_z f(z),\ d_{x=0}^2 u = d_z\{\overline{\phi(z)}|^2 d_z f(z)\},\ldots$$
$$d_{x=0}^n u = d_z^{n-1}\{\overline{\phi(z)}|^n d_z f(z)\},\ldots$$

And $u = u_{x=0} + d_{x=0}u\dfrac{x}{\underline{|1}} + d_{x=0}^2 u\dfrac{x^2}{\underline{|2}} + \ldots + d_{x=0}^n u\dfrac{x^n}{\underline{|n}} + \ldots$ (39);

$$\therefore\ f(y) = f(z) + \phi(z)\,d_z f(z)\frac{x}{\underline{|1}} + d_z\{\overline{\phi(z)}|^2 d_z f(z)\}\frac{x^2}{\underline{|2}}$$
$$+ d_z^2\{\overline{\phi(z)}|^3 d_z f(z)\}\frac{x^3}{\underline{|3}} + \ldots + d_z^{n-1}\{\overline{\phi(z)}|^n d_z f(z)\}\frac{x^n}{\underline{|n}} + \ldots$$

Cor. If $f(y) = y$, $f(z) = z$, $\therefore\ d_z f(z) = 1$; and therefore

$$y = z + \phi(z)\frac{x}{1} + d_z\overline{\phi(z)}|^2\frac{x^2}{\underline{|2}} + d_z^2\overline{\phi(z)}|^3\frac{x^3}{\underline{|3}} + \ldots + d_z^{n-1}\overline{\phi(z)}|^n\frac{x^n}{\underline{|n}} + \ldots$$

Since

$$d_z f(y) = d_z f(z) + d_z\{\phi(z)\,d_z f(z)\}\frac{x}{\underline{|1}} + d_z^2\{\overline{\phi(z)}|^2 d_z f(z)\}\frac{x^2}{\underline{|2}} + \ldots$$

and $d_z y = 1 + d_z\phi(z)\dfrac{x}{\underline{|1}} + d_z^2\overline{\phi(z)}|^2\dfrac{x^2}{\underline{|2}} + \ldots$

the law of the terms in the preceding theorems can be readily inferred from the expressions

$$d_z f(y) = \epsilon^{x d_z \phi(z)}\, d_z f(z).$$
$$d_z y \quad = \epsilon^{x d_z \phi(z)}.$$

51. Laplace's Theorem. To expand $u = f(y)$, in a series of ascending powers of x; where $y = \psi\{z + x\phi(y)\}$, and x and z are independent quantities.

Let $z + x\phi(y) = w$,

$$\therefore\ y = \psi(w),$$

$$d_x y = d_w\psi(w)\, d_x w = d_w\psi(w)\{\phi(y) + x\, d_y\phi(y)\, d_x y\},$$

$$d_z y = d_w\psi(w)\, d_z w = d_w\psi(w)\{1 + x\, d_y\phi(y)\, d_z y\};$$

$$\therefore\ d_x y\{1 - d_w\psi(w) x\, d_y\phi(y)\} = d_w\psi(w)\phi(y),$$

$$d_z y\{1 - d_w\psi(w) x\, d_y\phi(y)\} = d_w\psi(w);$$

$$\therefore\ d_x y = \phi(y)\, d_z y.$$

Whence, as in (50), we obtain

$$d_x^n u = d_z^{n-1}\{\overline{\phi(y)}|^n d_z u\}.$$

If we make $x = 0$ in the equation $y = \psi\{z + x\phi(y)\}$, y, $\phi(y)$, u, $d_z u$ become $\psi(z)$, $\phi\{\psi(z)\}$, $f\{\psi(z)\}$, $d_z f\{\psi(z)\}$ respectively. Therefore, by a process exactly like that in (50) we obtain

$$\begin{aligned} f(y) &= f[\psi(z)] + \phi[\psi(z)] d_z f[\psi(z)] \frac{x}{\lfloor\underline{1}} \\ &+ d_z\{\overline{\phi[\psi(z)]}|^2 d_z f[\psi(z)]\} \frac{x^2}{\lfloor\underline{2}} + \dots \\ &\dots + d_z^{n-1}\{\overline{\phi[\psi(z)]}|^n d_z f[\psi(z)]\} \frac{x^n}{\lfloor\underline{n}} + \dots \end{aligned}$$

Since $d_z f(y) = d_z f[\psi(z)] + d_z\{\phi[\psi(z)]\, d_z f[\psi(z)]\} \frac{x}{\lfloor\underline{1}} + \dots$ the law of the terms in Laplace's theorem may be inferred from the expression

$$d_z f(y) = \epsilon^{x d_z \phi[\psi(z)]}\, d_z f[\psi(z)].$$

52. To differentiate $f(y, z)$, y and z being functions of a third quantity x.

When x becomes $x+h$, y and z will become $y+\delta y$, and $z+\delta z$, where

$$\delta y = d_x y \frac{1}{h} + d_x^2 y \frac{h^2}{1 . 2} + \dots$$

$$\delta z = d_x z \frac{h}{1} + d_x^2 z \frac{h^2}{1 . 2} + \dots$$

Let $f(y, z) = u$; then, if y and z had been independent quantities, we should have had (46)

$$\begin{aligned} f(y + \delta y, z + \delta z) = u &+ d_y u \frac{\delta y}{1} + d_y^2 u \frac{(\delta y)^2}{1 . 2} + \dots \\ &+ d_z u \frac{\delta z}{1} + d_y d_z u \frac{\delta y}{1} \frac{\delta z}{1} + \dots \\ &+ d_z^2 u \frac{(\delta z)^2}{1 . 2} + \dots \\ &+ \dots \end{aligned}$$

Now, in order that this expansion of $f(y+\delta y, z+\delta z)$ may be applicable in the present case, where y and z are functions of the same quantity x, and, therefore, of each other, we must form the coefficients of δy, δz, ... by differentiating u as if no relation existed between y and z. Therefore, denoting the differential coefficients of u formed on this supposition, by $d_{(y)} u$, $d_{(z)} u$, ..., and substituting the values of δy, δz given above, we get

$$\begin{aligned} f(y + dy, z + dz) = u &+ d_{(y)} u . (d_x y \frac{h}{1} d_y^2 \frac{h^2}{1 . 2} + \dots) + \dots \\ &+ d_{(z)} u . (d_x z \frac{h}{1} d_x^2 z \frac{h^2}{1 . 2} + \dots) + \dots \\ &+ \dots \end{aligned}$$

But

$$f(y+\delta y,\ z+\delta z) = \text{value of } u \text{ when } x \text{ becomes } x+h$$

$$= u + d_x u \frac{h}{1} + d_x^2 u \frac{h^2}{1\,.\,2} + \ldots$$

Therefore, equating the coefficients of h in these two expansions,

$$d_x u = d_{(y)} u \,.\, d_x y + d_{(z)} u \,.\, d_x z.$$

Cor. Similarly, if $u = f(y_1, y_2, \ldots y_n)$, where $y_1, y_2, \ldots y_n$ are functions of x,

$$d_x u = d_{(y_1)} u \,.\, d_x y_1 + d_{(y_2)} u \,.\, d_x y_2 + \ldots + d_{(y_n)} u \,.\, d_x y_n.$$

53. Differentiation of implicit functions.

Let $f(x, y) = 0$ be an equation between x and y, in which y is considered as a function of x. Then, when x becomes $x+h$, y will become $y+k$, where

$$k = d_x y \frac{h}{1} + d_x^2 y \frac{h^2}{1\,.\,2} + \ldots$$

Now, if y had been independent of x, we should have had (writing u for $f(x, y)$)

$$\begin{aligned} f(x+h,\ y+k) = u &+ d_x u \frac{h}{1} + d_x^2 u \frac{h^2}{1\,.\,2} + \ldots \\ &+ d_y u \frac{k}{1} + d_x d_y u \frac{h}{1}\,\frac{k}{1} + \ldots \\ &+ d_y^2 u \frac{k^2}{1\,.\,2} + \ldots \\ &+ \ldots \end{aligned}$$

Therefore, denoting the differential coefficients of u, formed on the hypothesis that y is not a function of x, by $d_{(x)} u$, $d_{(y)} u, \ldots$

substituting the value of k given above, and observing that $f(x+h, y+k) = 0$, we get

$$0 = u + d_{(x)}u\frac{h}{1} + d_{(x)}^2 u\frac{h^2}{1.2} + \dots$$

$$+d_{(y)}u\left(d_x y\frac{h}{1} + d_x^2 y\frac{h^2}{1.2} + ..\right) + d_{(x)}d_{(y)}u\frac{k}{1}\left(d_x y\frac{h}{1} + d_x^2 y\frac{h^2}{1.2} + \right) + ..$$

$$+ d_{(y)}^2 u\frac{1}{1.2}\left(d_x y\frac{h}{1} + d_y^2\frac{h^2}{1.2} + \dots\right)^2 + \dots$$

$$+ \dots$$

And since this series $= 0$, whatever be the value of h, the coefficients of h must be separately equal to 0; therefore

$$d_{(x)}u + d_{(y)}u . d_x y = 0.$$

$$d_{(x)}^2 u + 2d_{(x)}d_{(y)}u . d_x y + d_{(y)}^2 u . (d_x y)^2 + d_{(y)}u . d_x^2 y = 0.$$

....

These equations are frequently termed the 1st, 2d, ... derived equations of $u = 0$, which is called the primitive equation.

The second of the above equations might have been derived from the first in the same way that the first was derived from $u = 0$. In like manner by successive derivations the 3d, 4th, ... derived equations may be formed.

It appears from (52) that the 1st, 2d, ... derived equations of $u = 0$ are identical with $d_x u = 0$, $d_x^2 u = 0$, ...

54. When $n+1$ variables are connected by n equations, n of the variables become implicit functions of the remaining one. Let $u_1 = 0$, $u_2 = 0, \dots u_n = 0$, be the equations between $x_1, x_2, \dots x_n, t$. And let t be considered the independent variable, and, therefore, $x_1, x_2, \dots x_n$ functions of t.

Since $u_1=0$, $u_2=0, \dots u_n=0$; $d_t u_1=0$, $d_t u_2=0, \dots d_t u_n=0$; therefore (52)

$$d_{(x_1)}u_1 . d_t x_1 + d_{(x_2)}u_1 . d_t x_2 + \dots + d_{(t)}u_1 = 0.$$

$$d_{(x_1)}u_2 . d_t x_1 + d_{(x_2)}u_2 . d_t x_2 + \dots + d_{(t)}u_2 = 0.$$

....

$$d_{(x_1)}u_n \; d_t x_1 + d_{(x_2)}u_n . d_t x_2 + \dots + d_{(t)}u_n = 0.$$

From which n equations the n quantities $d_t x_1$, $d_t x_2, \dots$ may be determined.

55. Differentiation of an implicit function of two variables.

Let $u=0$ be an equation between x, y, z; then, considering x and y as independent variables, z becomes a function of x and y.

$$\left.\begin{aligned} d_x u &= d_{(x)}u + d_{(z)}u . d_x z. \\ d_y u &= d_{(y)}u + d_{(z)}u . d_y z. \end{aligned}\right\} \quad (52).$$

But $d_x u = 0$, $d_y u = 0$;

$$\therefore \quad d_{(x)}u + d_{(z)}u . d_x z = 0. \qquad (\alpha)$$

$$d_{(y)}u + d_{(z)}u . d_y z = 0. \qquad (\beta)$$

Differentiating (α) with respect to x; (α) with respect to y, or (β) with respect to x; and (β) with respect to y, we obtain

$$d^2_{(x)}u + 2d_{(x)}d_{(z)}u . d_x z + d^2_{(z)}u . (d_x z)^2 + d_{(z)}u . d^2_x z = 0.$$

$$d_{(x)}d_{(y)}u + d_{(x)}d_{(z)}u . d_y z + d_{(y)}d_{(z)}u . d_x z + d^2_{(z)}u . d_x z \, d_y z + d_{(z)}u . d_x d_y z = 0.$$

$$d^2_{(y)}u + 2d_{(y)}d_{(z)}u . d_y z + d^2_{(z)}u . (d_y z)^2 + d_{(z)}u . d^2_y z = 0.$$

By differentiating the above equations with respect to x or y, equations may be formed involving $d^3_x z$, $d^2_x d_y z, \dots$ And in the same manner may be obtained the differential coefficients of z of a superior order.

56. If we had m equations $u_1 = 0$, $u_2 = 0, \ldots u_m = 0$, between $m+n$ quantities $x_1, x_2, \ldots x_m, t_1, t_2, \ldots t_n$, m of the quantities $x_1, x_2, \ldots$ might be considered as implicit functions of the n remaining quantities. Let each of the quantities $x_1, x_2, \ldots x_m$ be considered as an implicit function of $t_1, t_2, \ldots t_n$. Therefore, differentiating each of the equations $u_1=0$, $u_2=0, \ldots$ $u_m=0$, with respect to each of the quantities $t_1, t_2, \ldots t_n$, by the rule in (52), we get

$$d_{(t_1)}u_1 + d_{(x_1)}u_1 \cdot d_{t_1}x_1 + d_{(x_2)}u_1 \cdot d_{t_1}x_2 + \ldots = 0.$$

$$d_{(t_2)}u_1 + d_{(x_1)}u_1 \cdot d_{t_2}x_1 + d_{(x_2)}u \cdot d_{t_2}x_2 + \ldots = 0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$d_{(t_n)}u_1 + d_{(x_1)}u_1 \cdot d_{t_n}x_1 + d_{(x_2)}u_1 \cdot d_{t_n}x_2 + \ldots = 0.$$

$$d_{(t_1)}u_2 + d_{(x_1)}u_2 \cdot d_{t_1}x_1 + d_{(x_2)}u_2 \cdot d_{t_1}x_2 + \ldots = 0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From which mn equations the values of the mn differential coefficients $d_{t_1}x_1, d_{t_1}x_2, \ldots d_{t_2}x_1, d_{t_2}x_2, \ldots$ may be found.

57. Elimination of constants.

Let $u=0$ be an equation between x and y, in which y is considered as a function of x; and let u contain a constant a. It appears that in particular cases differentiation causes a to disappear (6). When, however, this does not happen, or when a is found in $d_x u$, a may be eliminated between $u=0$ and its derived equation $d_x u = 0$. The resulting equation expresses the value of $d_x y$ in terms of x and y, whatever be the value of a.

When a is above the first degree in u, the equation which results from the elimination of a will contain higher powers than the first of $d_x y$.

In like manner two constants may be eliminated between $u=0$, $d_x u=0$, $d_x^2 u=0$; three between $u=0$, $d_x u=0$, $d_x^2 u=0$, $d_x^3 u=0$. And, generally, n between $u=0$, $d_x u=0, \ldots d_x^n u=0$.

58. Elimination of functions.

Let v be any function of x and y; and, retaining the notation of (57), let u contain v^m, where m is a fraction; then $d_x u$ contains $d_x v^m = mv^{-1} d_x v . v^m$, and may contain v^m. Consequently the irrational function v^m may be eliminated between $u=0$ and $d_x u=0$. If u contained a second irrational function w^n, we might eliminate v^m, w^n between $u=0$, $d_x u=0$, $d_x^2 u=0$. And, generally, n irrational functions may be eliminated between $u=0$, $d_x u=0, \ldots d_x^n u=0$.

Let u contain a^v; then $d_x u$ contains $d_x a^v = \log_e a . d_x v . a^v$, and may contain a^v. Therefore a^v may be eliminated between $u=0$ and $d_x u=0$.

Let u contain $\log_e v$; then $d_x u$ contains v^{-1}, and may contain $\log_e v$, therefore $\log_e v$ may be eliminated between $u=0$, $d_x u=0$.

Lastly, let u contain $\phi(v)$, where $\phi(v)$ denotes any circular function of v, such as $\sin v$, $\tan v$, &c.; then $d_x u$ contains $d_x v \, d_v \phi(v)$, and may contain $\phi(v)$. Now $d_v \phi(v)$ can always be expressed in terms of $\phi(v)$. Therefore, after having substituted for $d_v \phi(v)$ its value in terms of $\phi(v)$, $\phi(v)$ may be eliminated between $u=0$ and $d_x u=0$.

As in the case of irrational functions, two functions of any of the above forms, may be eliminated between $u=0$, $d_x u=v$, $d_x^2 u=0$. And n such functions may be eliminated between $u=0$, $d_x u=0, \ldots d_x^n u=0$.

59. Let $u=0$ be an equation between x, y, z, in which z is considered as an implicit function of the independent variables x and y. Two quantities either constant or variable may be eliminated between $u=0$, and its derived equations $d_x u=0$, $d_y u=0$; three between $u=0$, $d_x u=0$, $d_y u=0$, $d_x^2 u=0$, $d_x d_y u=0$, $d_y^2 u=0$. And so on. In the same manner we may eliminate a function of x and y, the form of which is unknown. For, let t be a known function of x, y, s an unknown function of t; and let u contain s. Then, each of the quantities $d_x u$, $d_y u$

contain $d_t s$, and may contain s. And s and $d_t s$ may be eliminated between the equations $u = 0$, $d_x u = 0$, $d_y u = 0$. And thus we obtain a relation between x, y, z, $d_x z$, $d_y z$, wholly independent of the quantities eliminated in one case, and, of the form of the function of t, in the other.

We must not conclude that it is possible to eliminate two unknown functions, between $u = 0$, and its derived equations of the 1st and 2d order. For, let r be a known function of x, y, and q a function of r; and let u contain s and q. Then, each of the quantities $d_x u$, $d_y u$, will contain $d_t s$, $d_r q$, and may contain s, q; and each of the quantities $d_x^2 u$, $d_x d_y u$, $d_y^2 u$, will contain $d_t^2 s$, $d_r^2 q$, and may contain $d_t s$, s, $d_r q$, q. Hence, the number of functions may be equal to the number of equations, in which case the elimination is impossible. Particular cases do, however, occur in which the elimination of two functions can be effected.

All that has been said respecting the elimination of constants or functions between equations and their derivatives, applies equally to explicit functions.

60. Change of the Independent Variable. To express $d_x y$, $d_x^2 y, \dots$ in terms of $d_t x$, $d_t y$, $d_t^2 x$, $d_t^2 y, \dots$, x and y being functions of a third quantity t.

Let t become $t + e$; then x, y will become $x + h$, $y + k$ respectively, where

$$h = d_t x \frac{e}{1} + d_t^2 x \frac{e^2}{1 \cdot 2} + d_t^3 x \frac{e^3}{1 \cdot 2 \cdot 3} + \dots$$

$$k = d_t y \frac{e}{1} + d_t^2 y \frac{e^2}{1 \cdot 2} + d_t^3 y \frac{e^3}{1 \cdot 2 \cdot 3} + \dots$$

x and y are functions of the same quantity t, and, therefore, of each other, and when x becomes $x + h$, y becomes $y + k$,

$$\therefore\ k = d_x y \frac{h}{1} + d_x^2 y \frac{h^2}{1 \cdot 2} + d_x^3 y \frac{h^3}{1 \cdot 2 \cdot 3} + \dots$$

substitute for h its value given above, and we get

$$\begin{aligned} k = d_x y \left\{d_t x . e + \tfrac{1}{2} d_t^2 x . e^2 + \tfrac{1}{6} d_t^3 x . e^3 + \dots\right\} \\ + \tfrac{1}{2} d_x^2 y \left\{(d_t x)^2 . e^2 + d_t x d_t^2 x . e^3 + \dots\right\} \\ + \tfrac{1}{6} d_x^3 y \left\{(d_t x)^3 . e^3 + \dots\right\} \\ + \dots \end{aligned}$$

Therefore, equating the coefficients of like powers of e in the two expansions of k, we get

$$\begin{aligned} d_t y &= d_x y \, d_t x, \\ d_t^2 y &= d_x y \, d_t^2 x + d_t^2 y \, (d_t x)^2, \\ d_t^3 y &= d_x y \, d_t^3 x + 3 d_x^2 y \, d_t x \, d_t^2 x + d_x^3 y (d_t x)^3. \end{aligned}$$

....

$$\therefore \; d_x y = \frac{d_t y}{d_t x},$$

$$d_x^2 y = \frac{1}{(d_t x)^3} \left\{d_t x \, d_t^2 y - d_t y \, d_t^2 x\right\},$$

$$d_x^3 y = \frac{1}{(d_t x)^5} \left\{(d_t x)^2 d_t^3 y - 3 d_t x \, d_t^2 x \, d_t^2 y + 3 d_t y (d_t^2 x)^2 - d_t x \, d_t y \, d_t^3 x\right\}.$$

....

61. To express the relation between $d_x y$, $d_x^2 y$, ... and $d_y x$, $d_y^2 x$, ...

When x becomes $x+h$, let y become $y+k$; therefore, considering y as a function of x,

$$k = d_x y \frac{h}{1} + d_x^2 y \frac{h^2}{1.2} + d_x^3 y \frac{h^3}{1.2.3} + \dots$$

also, considering x as a function of y,

$$h = d_y x \frac{k}{1} + d_y^2 x \frac{k^2}{1.2} + d_y^3 x \frac{k^3}{1.2.3} + \dots$$

substitute the value of k found above, and we get

$$\begin{aligned} h = d_yx\,&\{d_xyh + \tfrac{1}{2}d_x^2yh^2 + \tfrac{1}{6}d_x^3yh^3 + \dots\} \\ &+ \tfrac{1}{2}d_y^2x\,\{(d_xy)^2h^2 + d_xyd_x^2yh^3 + \dots\} \\ &\qquad + \tfrac{1}{6}d_y^3\,\{(d_xy)^3h^3 + \dots\} \\ &\qquad\qquad + \dots \end{aligned}$$

Therefore, equating the coefficients of like powers of h on both sides of the equation,

$$\begin{aligned} 1 &= d_yxd_xy, \\ 0 &= d_yxd_x^2y + d_y^2x(d_xy)^2, \\ 0 &= d_yxd_x^3y + 3d_y^2xd_xyd_x^2y + d_y^3x(d_xy^3). \end{aligned}$$

. . . .

These results, as well as those of (60) might have been deduced from (5).

For, $1 = d_yxd_xy \quad (5)$,

$$\therefore\ d_xy = \frac{1}{d_yx},$$

$$d_x^2y = d_x\left(\frac{1}{d_yx}\right) = \frac{1}{d_yx}d_y\left(\frac{1}{d_yx}\right) = -\frac{d_y^2x}{(d_yx)^3}.$$

Also, writing y, x, t instead of u, y, x respectively, we get

$$d_ty = d_xyd_tx,$$

$$\therefore\ d_xy = \frac{d_ty}{d_tx},$$

$$d_x^2y = d_x\left(\frac{d_ty}{d_tx}\right) = \frac{1}{d_tx}d_t\left(\frac{d_ty}{d_tx}\right) = \frac{1}{(d_tx)^3}\{d_txd_t^2y - d_tyd_t^2x\}.$$

SECTION III.

PARTICULAR VALUES OF THE DIFFERENTIAL COEFFICIENT.

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62. VANISHING fractions.

Let u, v, the numerator and denominator of a fraction, be two functions of x that vanish when $x = a$. Then u, v must be of the form $p(x-a)^m$, $q(x-a)^n$, where p and q do not vanish when $x = a$, and

$$\frac{u}{v} = \frac{p(x-a)^m}{q(x-a)^n} = \frac{p}{q}(x-a)^{m-n}.$$

If $m > n$, $(x-a)^{m-n}$ vanishes when $x = a$;

$$\therefore \frac{u_{x=a}}{v_{x=a}} = 0.$$

If $m = n$, $(x-a)^{m-n} = 1$ when $x = a$;

$$\therefore \frac{u_{x=a}}{v_{x=a}} = \frac{p_{x=a}}{q_{x=a}}.$$

If $m < n$, $(x-a)^{m-n}$ becomes infinite when $x = a$,

$$\therefore \frac{u_{x=a}}{v_{x=a}} = \infty.$$

It appears then, that, when m and n are equal, the value of the fraction which takes the form $\frac{0}{0}$ when $x = a$, is finite and equal to the fraction $\frac{p_{x=a}}{q_{x=a}}$.

When m is a whole number $p_{x=a}$, $q_{x=a}$ may be found by differentiation.

$$u = p(x-a)^m,$$

$$d_x u = d_x p(x-a)^m + m(x-a)^{m-1}p,$$

$$d_x^2 u = d_x^2 p(x-a)^m + 2m d_x p(x-a)^{m-1} + m(m-1)(x-a)^{m-2}p.$$

. . . .

$$d_x^m u = d_x^m p(x-a)^m + \dots\dots\dots\dots\dots\dots + m.(m-1)\dots 3.2.1.p.$$

When $x=a$, every term except the last vanishes,

$$\therefore\ d_{x=a}^m u = m(m-1)\dots 3.2.1.p_{x=a}.$$

In the same manner we get

$$d_{x=a}^n v = n.(n-1)\dots 3.2.1.q_{x=a};$$

$$\therefore\ \text{when } n = m;$$

$$\frac{u_{x=a}}{v_{x=a}} = \frac{p_{x=a}}{q_{x=a}} = \frac{d_{x=a}^m u}{d_{x=a}^m v}.$$

63. When m is a fraction greater than k but less than $k+1$.

$$d_x^k u = \dots\dots + k(k-1)\dots 3.2.1.p(x-a)^{m-k},$$

$$d_x^{k+1} u = \dots\dots + (k+1)k\dots 3.2.1.p(x-a)^{m-k-1}.$$

When $x=a$, $(x-a)^{m-k}$ vanishes, and $(x-a)^{m-k-1}$ becomes infinite, therefore $d_{x=a}^k u = 0$, $d_{x=a}^{k+1} = \infty$.

Consequently, when either u or v involves a fractional power of $x-a$, the preceding rule for finding the value of $\frac{u}{v}$ when $x=0$, becomes inapplicable.

In this case write $a+h$ instead of k in u and v, and expand u and v in series of ascending powers of h by the ordinary algebraic methods. Then, if

$$u_{x=a+h} = Ah^{\alpha} + A'h^{\alpha'} + \dots$$

$$v_{x=a+h} = Bh^{\beta} + B'h^{\beta'} + \dots$$

$$\frac{u_{x=a+h}}{v_{x=a+h}} = \frac{Ah^{\alpha} + A'h^{\alpha'} + \dots}{Bh^{\beta} + B'h^{\beta'} + \dots}$$

$$= \frac{A\,.\,h^{\alpha-\beta} + A'h^{\alpha'-\beta} \dots}{B + B'h^{\beta'-\beta} + \dots}$$

If $\alpha > \beta$, $h^{\alpha-\beta}$ vanishes when $h=0$, $\therefore \dfrac{u_{x=a}}{v_{x=a}} = 0$.

If $\alpha = \beta$, $h^{\alpha-\beta} = 1$ when $h = 0$;

$$\therefore \frac{u_{x=a}}{v_{x=a}} = \frac{A}{B}.$$

If $\alpha < \beta$, $h^{\alpha-\beta}$ become infinite when $h=0$, therefore

$$\frac{u_{x=a}}{v_{x=a}} = \infty\,.$$

64. The fraction $\dfrac{u}{v}$ whose numerator and denominator become infinite when $x = a$, may be reduced to the form $\dfrac{0}{0}$.

For, $\dfrac{u}{v} = \dfrac{v^{-1}}{u^{-1}}$, $\therefore \dfrac{u_{x=a}}{v_{x=a}} = \dfrac{v^{-1}_{x=a}}{u^{-1}_{x=a}} = \dfrac{0}{0}$.

So also may the expression uv in which one factor becomes infinite and the other nothing when $x = a$. For, supposing $v_{x=a} = \infty$,

$$uv = \frac{u}{v^{-1}}, \quad \therefore u_{x=a}\,.\,v_{x=a} = \frac{u_{x=a}}{v^{-1}_{x=a}} = \frac{0}{0}.$$

65. Let $u=0$ be a rational equation between x and y; and let one of the values of y deduced from this equation involve the irrational expression $(x-a)\sqrt{F(x)}$. Then $d_x y$ will contain the expression $\sqrt{F(x)} + (x-a)\,d_x\sqrt{F(x)}$. Now, when $x = a$, the radical will disappear from y, but will remain in $d_x y$. Consequently $d_x y$ will have two values for each value of y.

Let b be one of the values of y when $x=a$; p, p' the two values of $d_x y$, and H, K the values of $d_{(x)}u$, $d_{(y)}u$, when $x=a$, $y=b$. Since u is a rational function of x, y; $d_{(x)}u$, $d_{(y)}u$ are rational functions of x, y; and, therefore, H, K are the only values of $d_{(x)}u$, $d_{(y)}u$, when $x=a$, $y=b$.

Then, since $d_{(x)}u + d_{(y)}u \cdot d_x y = 0$,

$$H + Kp = 0,$$

$$H + Kp' = 0;$$

$$\therefore\ K(p-p') = 0,$$

and $p-p'$ is finite, $\therefore\ K=0$, $\therefore\ H=0$;

and $d_x y$ takes the form $\frac{0}{0}$.

The values of $d_x y$ in this case are deducible from the second derived equation

$$d_{(x)}^2 u + 2d_{(x)}d_{(y)}u . d_x y + d_{(y)}^2 u . (d_x y)^2 + d_{(y)}u . d_x^2 y = 0.$$

$d_{(y)}u = K = 0$; therefore, L, M, N denoting the values of $d_{(x)}^2 u$, $d_{(x)}d_{(y)}u$, $d_{(y)}^2 u$, and p that of $d_x y$, we have

$$L + 2Mp + Np^2 = 0,$$

from which equation the two values of p may be deduced.

66. Let one of the values of y involve $(x-a)\{F(x)\}^{\frac{1}{3}}$; then, when $x=a$, $d_x y$ will have three values for each value of y. Let b be one of the values of y; p, p', p'' the corresponding values of $d_x y$. As in (65) L, M, N are the only corresponding values of $d_{(x)}^2 u$, $d_{(x)}d_{(y)}u$, $d_{(y)}^2 u$; therefore

$$L + 2Mp + Np^2 = 0,$$

$$L + 2Mp' + Np'^2 = 0,$$

$$L + 2Mp'' + Np''^2 = 0;$$

whence we obtain $L=0$, $M=0$, $N=0$.

In the derived equation of the third order the coefficients of d_x^2y, d_x^3y, are $3(d_{(x)}d_{(y)}u + d_{(y)}^2u.d_xy)$ and $d_{(y)}u$. These vanish when $x=a$, $y=b$, and the remaining terms form a cubic equation from which the three corresponding values of d_xy may be obtained.

In like manner, if y involve $(x-a)\{F(x)\}^{\frac{1}{n}}$, d_xy will have n values for each value of y, when $x=a$. And we must have recourse to the n^{th} derived equation in order to obtain the n corresponding values of d_xy.

Let y contain the expression $(x-a)^2.\{F(x)\}^{\frac{1}{n}}$; then $\{F(x)\}^{\frac{1}{n}}$ will be multiplied by $x-a$ in d_xy, but will occur disengaged from this factor in d_x^2y. Consequently when $x=a$, d_xy will have the same number of values as y, but d_x^2y will have n values for each value of y. And if y contain the expression $(x-a)^m.\{F(x)\}^{\frac{1}{n}}$, y, $d_xy, \ldots d_x^{m-1}y$ will have the same numbers of values when $x=a$, but d_x^my will have n values for each value of y.

In all these cases the differential coefficient that admits of a greater number of values than the preceding differential coefficient, appears under the form $\frac{0}{0}$, and its true value is to be obtained from one of the derived equations of a superior order.

67. Maxima and minima of functions of one variable.

Let $f(x)$ be any function of x; $f(x-h)$, $f(x+h)$ the same functions of $x-h$, $x+h$. Then, if the value of x be such that for any finite value of h however small, and for all inferior values of h greater than 0, $f(x-h)-f(x)$ and $f(x+h)-f(x)$ have the same sign, the value of $f(x)$ is said to be a maximum or a minimum according as $f(x-h)-f(x)$ and $f(x+h)-f(x)$ are both negative or both positive.

This definition merely implies that state of $f(x)$, arising from the substitution of a particular value for x, in which

$f(x)$ is greater than either of the quantities $f(x-h)$ and $f(x+h)$, in the case of a maximum; and less than $f(x-h)$ and $f(x+h)$, in the case of a minimum. Consequently $f(x)$ may admit of several maxima and minima, and some of its values may surpass a maximum, or fall short of a minimum, or even a maximum may be less than some of its minima.

68. To find the values of x, which render $f(x)$ a maximum or a minimum; and to distinguish a maximum from a minimum value of $f(x)$.

$$f(x-h) = f(x) - d_x f(x)\frac{h}{1} + d_x^2 f(x)\frac{h^2}{1.2} - d_x^3 f(x)\frac{h^3}{1.2.3} + \dots$$

$$f(x+h) = f(x) + d_x f(x)\frac{h}{1} + d_x^2 f(x)\frac{h^2}{1.2} + d_x^3 f(x)\frac{h^3}{1.2.3} + \dots$$

$$\therefore\ f(x-h) - f(x) = - d_x f(x)\frac{h}{1} + d_x^2 f(x)\frac{h^2}{1.2} - \dots$$

$$f(x+h) - f(x) = \quad d_x f(x)\frac{h}{1} + d_x^2 f(x)\frac{h^2}{1.2} + \dots$$

Since, by diminishing h, the first term of each of the above series may be made greater than the sum of all the terms that follow (42); and since the first term is negative in one series, and positive in the other, $f(x-h)-f(x)$ and $f(x+h)-f(x)$ will have different signs as long as $d_x f(x)$ is finite. But, by the definition, these quantities must have the same sign. Therefore, we must have $d_x f(x) = 0$, an equation from which may be deduced the values of x which render $f(x)$ a maximum or a minimum.

In this case then, we have

$$f(x-h) - f(x) = d_x^2 f(x)\frac{h^2}{1.2} - \dots$$

$$f(x+h) - f(x) = d_x^2 f(x)\frac{h^2}{1.2} + \dots$$

By diminishing h, the signs of $f(x-h)-f(x)$ and $f(x+h)-f(x)$ may be made the same as that of $d_x^2 f(x)$; and these quantities are negative in the case of a maximum, and positive in that of a minimum. Therefore $f(x)$ is a maximum or a minimum according as $d_x^2 f(x)$ is negative or positive.

69. If a value of x which renders $d_x f(x)=0$, also make $d_x^2 f(x)=0$, we shall have

$$f(x-h)-f(x) = -\,d_x^3 f(x)\frac{h^3}{1.2.3}+\dots$$

$$f(x+h)-f(x) = d_x^3 f(x)\frac{h^3}{1.2.3}+\dots$$

And, when h is made sufficiently small, $f(x-h)-f(x)$ and $f(x+h)-f(x)$ will have different signs; and therefore $f(x)$ will neither be a maximum nor a minimum.

If at the same time $d_x^3 f(x)=0$, $f(x)$ will be a maximum or a minimum, according as $d_x^4 f(x)$ is negative or positive. And, generally, when the first differential coefficient of $f(x)$ which does not vanish when a particular value is assigned to x, is of an even order, the corresponding value of $f(x)$ will be a maximum or a minimum according as the differential coefficient is negative or positive.

70. It may happen that the value of x which makes $f(x)$ a maximum or a minimum, renders it impossible to expand $f(x+h)$ by Taylor's theorem. Let a be such a value of x; and let

$$f(a+h)-f(a) = Ah^{\alpha}+Bh^{\beta}+\dots$$

α, β, ... being arranged in order of magnitude.

$$\therefore\ f(a-h)-f(a) = A(-h)^{\alpha}+B(-h)^{\beta}+\dots$$

provided none of the quantities α, β, ... be of the form $\dfrac{2m+1}{2n}$, for then either $f(a+h)$ or $f(a-h)$ would be impossible.

When a is of the form $\frac{2m}{2n+1}$; $f(a-h)-f(a)$ and $f(a+h)-f(a)$ may be made to have the same sign by diminishing h; and, therefore, when $x=a$, $f(x)$ is a maximum or a minimum according as A is negative or positive.

When a is of the form $\frac{2m+1}{2n+1}$; $f(a-h)-f(a)$ and $f(a+h)-f(a)$ have different signs, and therefore $f(a)$ is neither a maximum nor minimum value of $f(x)$. Therefore, since $d_{x=a}f(x)=\infty$, when a is less than 1, we must examine the values of x which make $d_x f(x)$ infinite, as well as those which make $d_x f(x)$ vanish. The same method may be extended to the case where a particular value of x makes one of the differential coefficients of $f(x)$ infinite, and all the preceding coefficients nothing.

When a is of the form $\frac{2m}{2n+1}$, and any of the succeeding exponents of the form $\frac{2m+1}{2n}$; one of the quantities $f(a-h)$, $f(a+h)$ will be impossible, and the other will have two values, both of which are less or greater than $f(a)$ according as A is negative or positive; $f(a)$ in this case is analogous to a maximum or minimum value of $f(x)$. It cannot, however, be so considered according to the definition of Art. 67, which requires the existence of both $f(a-h)$ and $f(a+h)$.

The operation of finding the values of x which render $f(x)$ a maximum or minimum, may frequently be abridged by observing that $\log_e f(x)$, and every positive power of $f(x)$ is a maximum or minimum when $f(x)$ is; and that every negative power of $f(x)$ is a minimum or maximum according as $f(x)$ is a maximum or minimum.

71. Let $f(x, y)$ be any function of x, y; $f(x+h, y+k)$ the same function of $x+h$, $y+k$. Then, if such values can be assigned to x and y, that, for any finite values of h and k, positive or negative, however small, and for all inferior values of h and k, $f(x+h, y+k)-f(x, y)$ may have the same sign;

$f(x, y)$ will be a maximum or a minimum according as that sign is negative or positive.

$$\begin{aligned} f(x+h, y+k) - f(x, y) = d_x f(x, y)\frac{h}{1} + \quad & d_x^2 f(x, y)\frac{h^2}{1.2} + \dots \\ + d_y f(x, y)\frac{k}{1} + & d_x d_y f(x, y)\frac{h}{1}.\frac{k}{1} + \dots \\ + \quad & d_y^2 f(x, y)\frac{k^2}{1.2} + \dots \\ & + \dots \end{aligned}$$

As long as $d_x f(x, y)$ and $d_y f(x, y)$ are finite, the sign of $f(x+h, y+k) - f(x, y)$, may be made to change with that of either h or k, by giving h and k particular values. Consequently, when $f(x, y)$ is a maximum or a minimum, we must have $d_x f(x, y) = 0$, $d_y f(x, y) = 0$. Among the values of x and y that satisfy these equations will be found those which make $f(x, y)$ a maximum or a minimum.

$$\begin{aligned} \therefore\ f(x+h, y+k) - f(x, y) &= d_x^2 f(x, y)\frac{h^2}{1.2} + d_x d_y f(x, y)\frac{h}{1}.\frac{k}{1} \\ &\qquad + d_y^2 f(x, y)\frac{k^2}{1.2} + \dots \\ &= \tfrac{1}{2}h^2.\{d_x^2 f(x, y) + 2 d_x d_y f(x, y)\, m + d_y^2 f(x, y)\, m^2\} + \dots \end{aligned}$$

if $k = mh$.

Now the sign of $f(x+h, y+k) - f(x, y)$ may be made to depend upon the sign of the coefficient of $\frac{1}{2}h^2$ by diminishing h. Hence, in order that $f(x, y)$ may be a maximum or a minimum, the sign of $d_x^2 f(x, y) + 2 d_x d_y f(x, y)\, m + d_y^2 fx, y)m^2$ must be incapable of changing, and, therefore, of becoming nothing, whatever be the value of m. Consequently the roots of the equation $0 = d_x^2 f(x, y) + 2 d_x d_y f(x, y)\, m + d_y^2 f(x, y)\, m^2$ must be impossible, and therefore

$$d_x^2 f(x, y) . d_y^2 f(x, y) > \{d_x d_y f(x, y)\}^2.$$

This condition cannot be fulfilled except $d_x^2 f(x, y)$, $d_y^2 f(x, y)$ have the same sign.

It appears then that the values of x and y deduced from the equations $d_x f(x, y) = 0$, $d_y f(x, y) = 0$, which make

$$d_x^2 f(x, y) \, . \, d_y^2 f(x, y) > \{d_x d_y f(x, y)\}^2,$$

render $f(x, y)$ a maximum or minimum, according as $d_x^2 f(x, y)$, $d_y^2 f(x, y)$ are negative or positive.

If the values of x, y that cause the coefficients of h, k to vanish, also cause the coefficients of h^2, hk, k^2 to vanish; the coefficients of h^3, $h^2 k$,... must each be equal to nothing, and the terms involving h^4, $h^3 k$,... must be incapable of changing their sign, in order that $f(x, y)$ may be a maximum or a minimum.

SECTION IV.

CURVES.

72. Let APQ (fig. 1.) be a curve traced upon a plane. Draw the straight lines OX, OY at right angles to each other in the same plane, PM parallel to OY. Then, the equation which expresses the relation between OM and PM, and which may be deduced from some known property of the curve, is called the equation to the curve AP.

73. A tangent to the curve AP at P, is the straight line through P, which limits the position of the chord joining P and any other point Q in the curve, when the arc PQ is indefinitely diminished.

To find the equation to a tangent to the curve AP at the point P.

Let X, Y be the co-ordinates of any point in the straight line PQ. Draw QN parallel to OY, and let $OM = x$, $MN = h$, $PM = y$; y is a function of x; and PM, QN are the values of y when those of x are OM, $OM + h$ respectively; therefore, denoting the values of $d_x y$, $d_x^2 y, \ldots$ by y', $y'', \ldots$ when $x = OM$,

$$QN = PM + y'h + \tfrac{1}{2}y''h^2 + \ldots$$

The equation to PQ (Hamilton's *Conic Sections*, 13.) is

$$Y - PM = \frac{QN - PM}{ON - OM}(X - OM),$$

$$= (y' + \tfrac{1}{2}y''h + \ldots)(X - OM).$$

And if PT be a tangent to AP at P, PQ coincides with PT when PQ, or MN, is diminished indefinitely; and the preceding equation becomes the equation to PT by making $h = 0$. Therefore, X and Y being the co-ordinates of any point in PT, the equation to PT is

$$Y - PM = y' (X - OM);$$

and the equation to a tangent at the point (x, y) is

$$Y - y = d_x y \, (X - x).$$

COR. 1. Hence $\tan PTX = d_x y$.

COR. 2. Let the tangent at P cut OX in T, and OY in V. Then, at T, $X = OT$, $Y = 0$, $\therefore\ OT = x - \dfrac{y}{d_x y}$; at V, $X = 0$, $Y = OV$, $\therefore\ OV = y - x d_x y$.

Hence, if in OX, OY we take $OT = x - \dfrac{y}{d_x y}$, $OV = y - x d_x y$, the straight line TV is a tangent to the curve at the point (x, y).

COR. 3. The subtangent is the distance of T from M

$$= OT - x = -\frac{y}{d_x y}.$$

74. Let rPR (fig. 2.) be a tangent to the curve qPQ at the point P; OX, OY rectangular axes; $Mn = MN$; qrn, PM, QRN parallel to OY; $AM = x$; $MN = h$, $MP = y$.

The equation to rPR is (73)

$$Y - y = d_x y \, (X - x);$$

$$\begin{aligned}
\therefore\ RN &= y + d_x y \,.\, h,\\
rn &= y - d_x y \,.\, h,\\
QN &= y + d_x y \,.\, h + \tfrac{1}{2} d_x^2 y \,.\, h^2 + \tfrac{1}{6} d_x^3 y \,.\, h^3 + \dots\\
qn &= y - d_x y \,.\, h + \tfrac{1}{2} d_x^2 y \,.\, h^2 - \tfrac{1}{6} d_x^3 y \,.\, h^3 + \dots\\
\therefore\ QN - RN &= \tfrac{1}{2} d_x^2 y \,.\, h^2 + \tfrac{1}{6} d_x^3 y \,.\, h^3 + \dots\\
qn - rn &= \tfrac{1}{2} d_x^2 y \,.\, h^2 - \tfrac{1}{6} d_x^3 y \,.\, h^3 + \dots
\end{aligned}$$

And, when h is sufficiently diminished, the sign of $\frac{1}{2}d_x^2y\,.\,h^2$ determines the sign of each of the above series; and therefore $QN-RN$ and $qn-rn$ have the same sign; or, qP and QP lie on the same side of the tangent qRP, which touches the curve at P, but does not cut it. When d_x^2y is positive; QN is greater than RN, and the curve PQ has its convexity turned toward the axis OX. When d_x^2y is negative; QN is less than RN, and the curve PQ has its concavity turned towards OX. If $d_x^2y=0$ at the point P,

$$QN-RN=\ \ \tfrac{1}{6}d_x^3y\,.\,h^3+\dots$$

$$qn-rn=-\tfrac{1}{6}d_x^3y\,.\,h^3+\dots$$

therefore $QN-RN$ and $qn-rn$ have different signs; and qP, PQ lie on opposite sides of the tangent which cuts the curve at P, as in fig. 3. P is in this case called "a point of inflexion."

75. When the point, P in which the tangent touches the curve, is indefinitely distant from 0; the tangent is called an asymptote.

If the values of OT, OV, or of both, do not become infinite, when either x or y, or both x and y are made infinite; the straight line VT is a tangent to the curve at a point indefinitely distant from O; and is, therefore, an asymptote to the curve.

Asymptotes may also be found in the following manner. Let x, y be the co-ordinates of any point in the curve. AP, fig. 4, referred to the axes OX, OY; and, if y can be expanded in a series of the form $ax+b+cx^{-1}+dx^{-2}+\dots$, the straight line whose equation is $Y=aX+b$ will be an asymptote.

Let ST be this straight line; and draw PSM parallel to OY. Then, if $OM=x$,

$$PM=ax+b+cx^{-1}+dx^{-2}+\dots$$

$$SM=ax+b;$$

$$\therefore\ SP=cx^{-1}+dx^{-2}+\dots$$

and therefore SP vanishes when x is indefinitely great, hence ST touches the curve AP at an indefinitely great distance from O, and is therefore an asymptote.

In like manner if x can be expanded in a series of the form

$$ay + b + cy^{-1} + dy^{-2} + \dots$$

the straight line $X = aY + b$ will be an asymptote.

76. A straight line drawn through the point in which a tangent touches a curve, at right angles to the tangent, is called a normal to the curve at that point.

To find the equation to PQ (fig. 1.), a normal to the curve AP at the point P.

The equation to the tangent PT is

$$Y - y = d_x y\,(X - x)$$

and PG is perpendicular to PT, and it passes through the point P, the co-ordinates of which are x, y; therefore, X, Y being the co-ordinates of any point in PG, the equation to PG is (Hamilton's *Conic Sections*. 18.)

$$(Y - y)\,d_x y + X - x = 0.$$

Cor. 1. Let PG cut OX in G, and OY in H. Then at the point G, $X = OG$, $Y = O$, therefore $OG = x + y\,d_x y$; at the point $H X = O$, $Y = OH$,

$$\therefore\ OH = y + \frac{x}{d_x y}.$$

Cor 2. The subnormal is the distance of G from M

$$= OG - x = y d_x y.$$

77. Let

$$a + a'x + a''x^2 + \dots$$
$$b + b'x + b''x^2 + \dots$$
$$c + c'x + c''x^2 + \dots$$

be three expressions of which the 2d is always intermediate in value between the other two, then, if $c = a$, $b = a$.

Since the value of the 2d is intermediate between the values of the other two,

$$a - b + (a' - b')\,x + (a'' - b'')\,x^2 + \dots \quad \text{and}$$
$$b - c + (b' - c')\,x + (b'' - c'')\,x^2 + \dots$$

have the same sign. As long as the first term of either series remains finite, its value may be made greater than the value of all the terms that follow by diminishing x (42), and, therefore, the sign of the series may be made to depend upon that of its first term. Hence, as long as $a - b$ and $b - c$ are finite, they must have the same sign. But, if $c = a$, $a - b$ and $b - c$ have different signs; therefore we must have $a - b = 0$; or, $b = a$.

78. To find the differential coefficient of the arc of a plane curve.

Let A be a fixed point in the curve APQ (fig. 5); OX, OY rectangular axes; PM, QN parallel to OY; PR a tangent to AP at P; PT parallel to OX. Let $OM = x$, $MN = h$, $PM = y$, $AP = s$. The place of P depends upon that of M, therefore s is a function of x; and PQ is the quantity by which s is increased when x becomes $x + h$,

$$\therefore\ PQ = d_x s \,.\, h + \tfrac{1}{2} d_x^2 s \,.\, h^2 + \dots$$

$$RQ = \mathrm{Q}h^2, \text{ if } \mathrm{Q}h^2 = \tfrac{1}{2} d_x^2 y \,.\, h^2 + \dots,$$

$$TQ = ON - PM = d_x y \,.\, h + \mathrm{Q}h^2,\ \ TR = d_x y \,.\, h\,;$$

$$\therefore\ PR + QR = \sqrt{\{PT^2 + TR^2\}} + QR = h\sqrt{\{1 + (d_x y)^2\}} + \mathrm{Q}h^2.$$

$$\text{Chord } PQ = \sqrt{\{PT^2 + TQ^2\}} = \sqrt{\{h^2 + (d_x y \,.\, h + \mathrm{Q} \,.\, h^2)^2\}}$$
$$= h\sqrt{\{1 + (d_x y)^2\}} + \{1 + (d_x y)^2\}^{-\frac{1}{2}} d_x y \,.\, \mathrm{Q}h^2 + \dots$$

Now the arc PQ, which between P and Q always bends the same way, is manifestly greater than the chord PQ, and less than $PR + RQ$; or, substituting the values of PR, &c. and dividing by h,

$$d_x s + \tfrac{1}{2} d_x^2 s \, . \, h + \ldots \text{ lies between}$$

$$\sqrt{\{1 + (d_x y)^2\}} + \{1 + (d_x y)^2\}^{-\frac{1}{2}} d_x y \, . \, \mathrm{Q} h \text{ and}$$

$$\sqrt{\{1 + (d_x y)^2\}} + \mathrm{Q} h;$$

and these expressions have the same first term, therefore (77)

$$d_x s = \sqrt{\{1 + (d_x y)^2\}}.$$

79. To find the differential coefficient of the area included between the curve AP (fig. 5.) the axis OX, and two straight lines AB, PM parallel to OY.

Draw QT perpendicular to TPM; and let the area $APMB = A$. The magnitude of this area depends upon the place of M, therefore A is a function of x; and $PQMN$ is the quantity by which A is increased when x becomes $x + h$,

$$\therefore \text{ area } PQMN = d_x A \, . \, h + \tfrac{1}{2} d_x^2 A \, . \, h^2 + \ldots$$

$$\text{also area } PN = PM \, . \, MN = y h,$$

$$\text{and area } QM = QN \, . \, MN = (y + d_x y \, . \, h + \tfrac{1}{2} d_x^2 y \, . \, h^2 + \ldots) \, h.$$

And the area $PQNM$ lies between the area PN and the area QM; therefore, substituting the values of these areas, and dividing by h,

$$d_x A + \tfrac{1}{2} d_x^2 A \, . \, h + \ldots \text{ lies between}$$

$$y \text{ and } y + d_x y \, . \, h + \ldots;$$

and these expressions have the same first term, therefore (77)

$$d_x A = y.$$

80. To find the differential coefficient of the volume of the solid generated by the revolution of $APMB$ (fig. 5.) round OX.

Let V be the volume of the solid generated by the revolution of $APMB$ round OX. Then, since V is a function of x, and the solid generated by the revolution of $PQNM$ is the quantity by which V is increased when x becomes $x + h$,

$$\text{solid generated by } PQNM = d_x V . h + \tfrac{1}{2} d_x^2 V . h^2 + \dots$$

Solid generated by

$$PN = (\text{area circle rad. } PM) . MN$$

$$= \pi . PM^2 . MN \text{ (Hymers's \textit{Integral Calculus}. 98.)}$$

$$= \pi y^2 h.$$

$$\text{Solid generated by } QM = \pi . QM^2 . MN$$

$$= \pi (y + d_x y . h + \dots)^2 h = \pi (y^2 + 2 y d_x y . h + \dots) h.$$

And the solid generated by $PQNM$ is greater than the solid generated by PN, and less than the solid generated by QM. Therefore $d_x V + \frac{1}{2} d_x^2 V . h + \dots$ is greater than πy^2, and less than $\pi (y^2 + 2 y d_x y . h + \dots)$; therefore (77)

$$d_x V = \pi y^2.$$

81. To find the differential coefficient of the area of the surface generated by the revolution of the curve AP (fig. 5.) round OX.

Let S be the area of the surface generated by the revolution of AP round OX. S is a function of x, and the surface generated by PQ is the quantity by which S is increased when x becomes $x + h$; therefore

$$\text{surface generated by } PQ = d_x S . h + \tfrac{1}{2} d_x^2 S . h^2 + \dots$$

The convex surface of a cylinder whose axis is equal to the length of the arc PQ, and whose radius is $PM = 2\pi . PM . PQ$ (Hymers's *Integral Calculus*. 98.)

$$= 2\pi y (d_x s . h + \tfrac{1}{2} d_x^2 s . h^2 + \dots)$$

and the convex surface of the cylinder whose axis is equal to PQ, and

radius $QN = 2\pi \,.\, QN \,.\, PQ$

$$= 2\pi\,(y + d_x y \,.\, h + \ldots)\,(d_x s \,.\, h + \tfrac{1}{2} d_x^2 s \,.\, h^2 + \ldots)$$

$$= 2\pi\,(y \,.\, d_x s \,.\, h + d_x y\, d_x s \,.\, h^2 + \tfrac{1}{2} y\, d_x^2 s \,.\, h^2 + \ldots)$$

And the surface generated by PQ is manifestly greater than that of one of the cylinders, and less than that of the other; therefore, dividing by h, $d_x S + \frac{1}{2} d_x^2 S \,.\, h + \ldots$ lies between

$$2\pi\,(y\, d_x s + \tfrac{1}{2} y\, d_x^2 s \,.\, h + \ldots)$$

and $2\pi\,(y\, d_x s + d_x y \,.\, d_x s \,.\, h + \frac{1}{2} y\, d_x^2 s \,.\, h + \ldots)$;

and these expressions have the same first term, ∴ (77)

$$d_x S = 2\pi y\, d_x s.$$

and $d_x s = \sqrt{\{1 + (d_x y)^2\}}$ (78);

$$\therefore\ d_x S = 2\pi y \sqrt{\{1 + (d_x y)^2\}}.$$

82. To find the radius of curvature, and the co-ordinates of the centre of curvature of a curve at any point.

Let P be any point in the curve AP (fig. 6.) Q, Q' any other points in the curve; then, if a circle be described through the three points P, Q, Q', the circle which is the limit to which the circle PQQ' approaches when PQ, PQ' are diminished indefinitely, is called the circle of curvature of the curve AP at the point P.

The radius of this circle is called the radius of curvature, and its centre is called the centre of curvature of the curve at the point P.

Let OX, OY be rectangular axes, PM, QN, $Q'N'$ parallel to OY; $MN = h$, $MN' = h'$; and let x, y be the co-ordinates of any point in the curve AP; X, Y those of any point in the circle PQQ'. PM is the value of y and of Y when x and X

are each equal to OM; and QM is the value of y and of Y when x and X are each equal to $OM + h$. Therefore, denoting the values of $d_x y$, $d_x^2 y, \ldots$ $d_X Y$, $d_X^2 Y, \ldots$ by $y', y'', \ldots$ $Y', Y'', \ldots$, when x and X are each equal to OM,

$$QN = PM + y'h + \tfrac{1}{2}y''h^2 + \tfrac{1}{6}y'''h^3 + \ldots$$

$$QN = PM + Y'h + \tfrac{1}{2}Y''h^2 + \tfrac{1}{6}Y'''h^3 + \ldots$$

therefore, subtracting and dividing by h,

$$0 = Y' - y' + \tfrac{1}{2}(Y'' - y'')h + \tfrac{1}{6}(Y''' - y''')h^2 + \ldots$$

In the same manner we get

$$0 = Y' - y' + \tfrac{1}{2}(Y'' - y'')h' + \tfrac{1}{6}(Y''' - y''')h'^2 + \ldots$$

therefore, subtracting and dividing by $\tfrac{1}{2}(h' - h)$,

$$0 = Y'' - y'' + \tfrac{1}{3}(Y''' - y''')(h' + h) + \ldots$$

Now when PQ, PQ' vanish, the circle PQQ' becomes the circle of curvature at P; therefore if we make h and h' each equal to nothing X, Y become the co-ordinates of any point in the circle of curvature, and we get

$$Y' = y', \quad Y'' = y''.$$

Hence, the circle of curvature at the point (x, y), is determined by the conditions $X = x$, $Y = y$, $d_X Y = d_x y$, $d_X^2 Y = d_x^2 y$.

Let a, b be the co-ordinates of the centre of the circle of curvature, r its radius; then its equation will be (Hamilton's *Conic Sections.* 33.)

$$r^2 = (Y - b)^2 + (X - a)^2,$$

$$\therefore\ 0 = (Y - b)\, d_X Y + X - a,$$

$$0 = (Y - b)\, d_X^2 Y + (d_X Y)^2 + 1.$$

But when $X = x$, $Y = y$, $d_X Y = d_x y$, $d_X^2 Y = d_x^2 y$; therefore

$$r^2 = (y - b)^2 + (x - a)^2 \ldots\ldots\ldots\ldots (\alpha),$$

$$0 = (y - b)\, d_x y + x - a \ldots\ldots\ldots (\beta),$$

$$0 = (y - b)\, d_x^2 y + (d_x y)^2 + 1 \ldots (\gamma);$$

$$\therefore\ y - b = -\frac{1}{d_x^2 y}\left\{1 + (d_x y)^2\right\},$$

$$x - a = \frac{d_x y}{d_x^2 y}\left\{1 + (d_x y)^2\right\},$$

$$r = -\frac{1}{d_x^2 y}\left\{1 + (d_x y)^2\right\}^{\frac{3}{2}}.$$

The expression for r appears with a double sign, it is customary however to use the negative value, and then, when the centre of curvature and the point in which a normal at P cuts the axis of x, lie on the same side of the curve, $d_x^2 y$ is negative, and, therefore, r is positive; when they lie on different sides of the curve, $d_x^2 y$ is positive, and, therefore, r is negative.

Let A be a given point in the curve, P a point whose co-ordinates are x, y; and let the arc $AP = s$. Then

$$(d_x s)^2 = 1 + (d_x y)^2;$$

$$\therefore\ r = -\frac{(d_x s)^3}{d_x^2 y}.$$

83. Let x and y be functions of a third quantity t. Then, $d_x y = \dfrac{d_t y}{d_t x}$,

$$d_x^2 y = \frac{1}{(d_t x)^3}\left\{d_t x \,.\, d_t^2 y - d_t y\, d_t^2 x\right\} \quad (60);$$

$$\therefore\ r = \frac{\left\{(d_t x)^2 + (d_t y)^2\right\}^{\frac{3}{2}}}{d_t y \,.\, d_t^2 x - d_t x \,.\, d_t^2 y}.$$

Considering x and y as functions of s, we get

$$r = \frac{\left\{(d_s x)^2 + (d_s y)^2\right\}^{\frac{3}{2}}}{d_s y \,.\, d_s^2 x - d_s x \,.\, d_s^2 y},$$

$$(d_x s)^2 = 1 + (d_x y)^2\ (78) = 1 + \frac{(d_s y)^2}{(d_s x)^2},$$

and $d_x s \,.\, d_s x = 1$ (61), $\therefore\ 1 = (d_s x)^2 + (d_s y)^2$,

$$\therefore\ 0 = d_s x \,.\, d_s^2 x + d_s y \,.\, d_s^2 y;$$

$$\therefore\ r = \frac{1}{d_s y \,.\, d_s^2 x - d_s x \,.\, d_s^2 y}$$

$$= \frac{d_s y}{d_s^2 x}$$

$$= -\frac{d_s x}{d_s^2 y}.$$

$$\frac{1}{r^2} = \left\{d_s y \,.\, d_s^2 x - d_s x \,.\, d_s^2 y\right\}^2$$

$$= (d_s y)^2 (d_s^2 x)^2 - 2 d_s y \,.\, d_s^2 x \,.\, d_s x \,.\, d_s^2 y + (d_s x)^2 (d_s^2 y)^2,$$

$$0 = \left\{d_s x \,.\, d_s^2 x + d_s y \,.\, d_s^2 y\right\}^2$$

$$= (d_s x)^2 (d_s^2 x)^2 + 2 d_s x \,.\, d_s^2 x \,.\, d_s y \,.\, d_s^2 y + (d_s y)^2 (d_s^2 y)^2;$$

$$\therefore\ \frac{1}{r^2} = \left\{(d_s x)^2 + (d_s y)^2\right\} . \left\{(d_s^2 x)^2 + (d_s^2 y)^2\right\} = (d_s^2 x)^2 + (d_s^2 y)^2;$$

$$\therefore\ r = \frac{1}{\sqrt{\left\{(d_s^2 x)^2 + (d_s^2 y)^2\right\}}}.$$

Let n be the angle between a normal to the curve at P and the axis of x. Then $\tan n = -\dfrac{1}{d_x y}$,

$$\therefore\ n = -\operatorname{cotan}^{-1} d_x y,\quad \therefore\ d_x n = \frac{d_x^2 y}{1 + (d_x y)^2},$$

$$\therefore\ r = -\frac{d_x s}{d_x n}.$$

84. We obtain from (β) (82) $(b-y)\, d_x y + a - x = 0$. This is the equation to a normal to the curve at the point P (76), having a and b in the place of X and Y. Hence the centre of the circle of curvature at any point lies in a normal to the curve at that point. Hence also the curve and the circle of

curvature at any point in it, have a common tangent at that point. This might have been inferred from the equations

$$X = x, \quad Y = y, \quad d_X Y = d_x y.$$

85. The evolute of a curve is the locus of the centre of curvature.

a, b are the co-ordinates of the centre of curvature at the point (x, y), and, therefore, of a point in the evolute. Hence, if we eliminate x, y between the equations

$$y - b = -\frac{1}{d_x^2 y}\{1 + (d_x y)^2\}$$

$$x - a = \frac{d_x y}{d_x^2 y}\{1 + (d_x y)^2\},$$

and the equation to the curve, the resulting equation between a, b, which is independent of x, will be the equation to the evolute (72).

86. Differentiating the equation

$$0 = (y - b)\, d_x y + x - a \quad (82) \text{ we get}$$

$$0 = (y - b)\, d_x^2 y + (d_x y)^2 - d_x b \,.\, d_x y + 1 - d_x a.$$

But $0 = (y - b)\, d_x^2 y + (d_x y)^2 + 1 \quad (82)$;

therefore, subtracting the upper equation from the lower,

$$0 = d_x b \,.\, d_x y + d_x a,$$

$$\therefore\ (y - b)\, d_x a = (x - a)\, d_x b,$$

$$\therefore\ y - b = d_a b\, (x - a) \qquad (\alpha);$$

$$\text{and } (b - y)\, d_x y + a - x = 0 \quad (\beta).$$

It appears from equation (α), that (x, y) is a point in a tangent to the evolute at the point (a, b); and from equation (β), that (a, b) is a point in a normal to the curve at the point (x, y). Hence a tangent to the evolute is a normal to the curve.

87. Differentiating the equation

$$r^2 = (y-b)^2 + (x-b)^2 \text{ (82) we get}$$

$$r d_x r = (y-b)\, d_x y - (y-b)\, d_x b + x - a - (x-a)\, d_x a$$

$$\text{But } 0 = (y-b)\, d_x y + x - a;$$

$$\therefore \quad -r d_x r = (y-b)\, d_x b + (x-a)\, d_x a,$$

$$= \left\{(d_x a)^2 + (d_x b)^2\right\} \frac{x-a}{d_x a}.$$

$$r^2 = (y-b)^2 + (x-a)^2 = \left\{(d_x a)^2 + (d_x b)^2\right\} \left(\frac{x-a}{d_x a}\right)^2;$$

$$\therefore \quad -d_x r = \pm \sqrt{\left\{(d_x a)^2 + (d_x b)^2\right\}}.$$

Let A', P' (fig. 6.) be the centres of curvature of the curve AP at the points A, P respectively; and let $A'P' = s'$; then

$$d_x s' = \sqrt{\left\{(d_x a)^2 + (d_x b)^2\right\}};$$

$$\therefore \quad d_x r \pm d_x s' = 0.$$

This is the equation which would have been obtained by differentiating the equation $r \pm s = C$, where C is constant (6). Taking the upper sign, as the present case manifestly requires, we get $A'P' + P'P =$ constant. Hence if a string $DP'P$, of invariable length, one end of which is fastened at D, be wound round the evolute $A'P'$, and then unwound, the extremity P will describe the curve AP. It is on account of this property, that $A'P'$ is called the evolute of AP, and AP is called the involute of $A'P'$.

88. Spirals. Let P (fig. 7.) be any point in the curve AP; SX, SV rectangular axes; PT a tangent to the curve at P, cutting SX, SV in T, V. Draw PM perpendicular to SX, SY perpendicular to PT; and let $SM = x$, $PM = y$, $PSX = \theta$, $SP = \rho$, $\rho u = 1$, $SY = p$, $AP = s$, r the radius of curvature at P, and $2q$ the chord of curvature through S. The equation which expresses the relation between ρ and θ, or between ρ and p is called the polar equation to the curve AP.

89. To find p and the angle SPY in terms of u and $d_\theta u$.

$$x = \rho \cos\theta = \frac{\cos\theta}{u},$$

$$y = \rho \sin\theta = \frac{\sin\theta}{u};$$

$$d_\theta x = -\frac{\sin\theta}{u} - \frac{\cos\theta}{u^2} d_\theta u,$$

$$d_\theta y = \frac{\cos\theta}{u} - \frac{\sin\theta}{u^2} d_\theta u;$$

$$\frac{d_\theta y}{d_\theta x} = \frac{\tan\theta - \dfrac{u}{d_\theta u}}{1 + \tan\theta \dfrac{u}{d_\theta u}}$$

$$= \tan\left\{\theta - \tan^{-1}\frac{u}{d_\theta u}\right\};$$

$$\therefore\ \tan^{-1}\frac{d_\theta y}{d_\theta x} = \theta - \tan^{-1}\frac{u}{d_\theta u}.$$

$$\frac{d_\theta y}{d_\theta x} = d_x y = \tan PTX.$$

$$SPY = PTX - PSX$$

$$= \tan^{-1}\frac{d_\theta y}{d_\theta x} - \theta = -\tan^{-1}\frac{u}{d_\theta u};$$

$$\therefore\ \tan SPY = -\frac{u}{d_\theta u}.$$

$$\left(\frac{SP}{SY}\right)^2 = (\text{cosecant } SPY)^2$$

$$= 1 + (\text{cotan } SPY)^2;$$

$$\therefore \quad \frac{1}{u^2 p^2} = 1 + \frac{(d_\theta u)^2}{u^2};$$

$$\therefore \quad \frac{1}{p^2} = u^2 + (d_\theta u)^2.$$

90. To express $d_\theta s$ in terms of u, $d_\theta u$, &c.

$$\begin{aligned}(d_\theta s)^2 &= (d_\theta x)^2 + (d_\theta y)^2 \\ &= \frac{1}{u^4}\left\{u^2 + (d_\theta u)^2\right\} \\ &= \frac{1}{u^4 p^2}.\end{aligned}$$

91. To express r in terms of u, $d_\theta u$, &c.

Differentiate the equation

$$\tan^{-1}\frac{d_\theta y}{d_\theta x} = \theta - \tan^{-1}\frac{u}{d_\theta u} \quad (89);$$

with respect to θ, and we get

$$\begin{aligned}\frac{d_\theta x \,.\, d_\theta^2 y - d_\theta y \,.\, d_\theta^2 x}{(d_\theta x)^2 + (d_\theta y)^2} &= 1 - \frac{(d_\theta u)^2 - u\, d_\theta^2 u}{u^2 + (d_\theta u)^2}, \\ &= \frac{u^2 + u\, d_\theta^2 u}{u^2 + (d_\theta u)^2}.\end{aligned}$$

But $(d_\theta x)^2 + (d_\theta y)^2 = \dfrac{1}{u^4}\left\{u^2 + (d_\theta u)^2\right\}$;

$$\therefore \quad d_\theta x \,.\, d_\theta^2 y - d_\theta y \,.\, d_\theta^2 y = \frac{1}{u^3}\left\{u + d_\theta^2 u\right\},$$

and therefore (83) $r = \dfrac{1}{u^3}\,\dfrac{\left\{u^2 + (d_\theta u)^2\right\}^{\frac{3}{2}}}{u + d_\theta^2 u}$

$$= \frac{u^3 (d_\theta s)^3}{u + d_\theta^2 u}.$$

$$q = r \sin SPY$$

$$= \frac{1}{u^2} \frac{u^2 + (d_\theta u)^2}{u + d_\theta^2 u}$$

$$= \frac{u^2 (d_\theta s)^2}{u + d_\theta^2 u}.$$

92. Let θ be a function of t. Then

$$d_t \frac{d_t y}{d_t x} = d_t \frac{d_\theta y}{d_\theta x} = d_t\theta \,.\, d_\theta \frac{d_\theta y}{d_\theta x},$$

$$\therefore \frac{1}{(d_t x)^2} \{d_t x \,.\, d_t^2 y - d_t y \,.\, d_t^2 x\} = \frac{d_t\theta}{(d_\theta x)^2} \{d_\theta x \,.\, d_\theta^2 y - d_\theta y \,.\, d_\theta^2 x\};$$

$$\text{and } d_t x = d_\theta x \,.\, d_t\theta;$$

$$\therefore d_t x \,.\, d_t^2 y - d_t y \,.\, d_t^2 x = (d_t\theta)^3 \{d_\theta x \,.\, d_\theta^2 y - d_\theta y \,.\, d_\theta^2 x\}.$$

This equation is of use in the solution of certain Dynamical problems.

93. To find the differential coefficient with respect to θ, of the area bounded by the curve AP and the radii AS, PS.

$$ASP = ASB + PSM - SBPM,$$

$$\therefore d_\theta (ASP) = d_\theta (PSM) - d_\theta (SBPM)$$

$$= d_\theta (\tfrac{1}{2} xy) - y\, d_\theta x$$

$$= \tfrac{1}{2} x\, d_\theta y + \tfrac{1}{2} y\, d_\theta x - y\, d_\theta x$$

$$\therefore 2 d_\theta (ASP) = x\, d_\theta y - y\, d_\theta x,$$

$$\frac{y}{x} = \tan\theta;$$

$$\therefore\ \frac{1}{x^2}(x\,d_\theta y - y\,d_\theta x) = (\text{secant } \theta)^2;$$

$$\therefore\ x\,d_\theta y - y\,d_\theta x = (x \text{ secant } \theta)^2 = \frac{1}{u^2};$$

$$\therefore\ 2\,d_\theta(ASP) = \frac{1}{u^2}.$$

$$= p\,d_\theta s \quad (90).$$

94. $u = \dfrac{1}{\rho}$, $\therefore\ d_\theta u = -\dfrac{d_\theta \rho}{\rho^2}$, $d_\theta^2 u = 2\dfrac{(d_\theta \rho)^2}{\rho^3} - \dfrac{d_\theta^2 \rho}{\rho^2}$;

therefore, substituting these values of u, $d_\theta u$, $d_\theta^2 u$ in the preceding equations, we get

$$\tan SPY = \frac{\rho}{d_\theta \rho}.$$

$$\frac{\rho^4}{p^2} = \rho^2 + (d_\theta \rho)^2,$$

$$(d_\theta s)^2 = \rho^2 + (d_\theta \rho)^2,$$

$$= \frac{\rho^4}{p^2}.$$

$$(d_\rho s)^2 = \frac{(d_\theta s)^2}{(d_\theta \rho)^2} = \frac{\rho^2}{\rho^2 - p^2}.$$

$$2\,d_\theta(ASP) = \rho^2.$$

$$r = \frac{\{\rho^2 + (d_\theta \rho)^2\}^{\frac{3}{2}}}{\rho^2 + 2(d_\theta \rho)^2 - \rho\, d_\theta^2 \rho}.$$

$$q = \frac{\rho\,\{\rho^2 + (d_\theta \rho)^2\}}{\rho^2 + 2(d_\theta \rho)^2 - \rho\, d_\theta^2 \rho}.$$

95. To express r in terms of ρ, p, $d_p\rho$.

The equation to TP is

$$Y - y = d_x y\,(X - x) \quad (73).$$

$$\text{At } V,\ X = 0,\ \therefore\ SV = y - x d_x y.$$

$$p = SV.\cos STP; \quad \text{and } \tan STP = -\,d_x y;$$

$$\therefore\ p = \frac{y - x d_x y}{\sqrt{\{1 + (d_x y)^2\}}}.$$

$$\rho^2 = x^2 + y^2,$$

$$\therefore\ \rho d_x \rho = x + y d_x y.$$

$$d_x p = -\frac{d_x^2 y}{\{1 + (d_x y)^2\}^{\frac{3}{2}}}\{x + y d_x y\}$$

$$= \frac{\rho d_x \rho}{r};$$

$$\therefore\ r = \rho\frac{d_x \rho}{d_x p} = \rho d_p \rho.$$

$$q = r \sin SPY$$

$$= r\frac{p}{\rho}$$

$$= p\,d_p\,\rho.$$

96. Let P' be the centre of curvature of the curve at the point P; therefore PP' is perpendicular to PY, and touches the evolute $A'P'$ at P (86). Draw SY' perpendicular to PP'; and let $SP' = \rho$, $SY' = p'$.

$$SY'^2 = PY^2 = SP^2 - SY^2,$$

$$SP'^2 - SY'^2 = P'Y'^2 = (PP' - SY)^2;$$

$$\text{and } PP' = \rho d_\rho p \quad (95);$$

$$\therefore\ p'^2 = \rho^2 - p^2,$$

$$\rho'^2 - p'^2 = (\rho d_\rho p - p)^2.$$

The polar equation to *A′P′*, the evolute of *AP*, may be obtained by eliminating ρ and p between these two equations and the equation to the curve. In the same manner, having given the equation between ρ' and p', or the polar equation to the curve *A′P′*, we may find the polar equation to *AP*, the involute of *A′P′*.

97. If ρ become infinite, while θ and p remain finite, the spiral will have an asymptote, the position of which is determined by the values of θ and p. If for any finite value (a) of ρ, θ become infinite the spiral will have an asymptotic circle, the radius of which is a.

98. Let YZ be a tangent to the curve traced out by Y. Draw SZ perpendicular to YZ; and let $ZSX = \phi$. Then $PSY = \theta - \phi$;

$$\cos(\theta - \phi) = \frac{p}{\rho}; \quad \log_e \cos(\theta - \phi) = \log_e p - \log_e \rho.$$

$$\therefore\ -\tan(\theta - \phi)(1 - d_\theta \phi) = \frac{d_\theta p}{p} - \frac{d_\theta \rho}{\rho};$$

$$\text{and } \tan(\theta - \phi) = \cot SPY = \frac{d_\theta \rho}{\rho} \quad (94);$$

$$\therefore\ \tan(\theta - \phi)\, d_\theta \phi = \frac{d_\theta p}{p} = \frac{d_\phi p}{p}\, d_\theta \phi;$$

$$\therefore\ \tan(\theta - \phi) = \frac{d_\phi p}{p};$$

$$\therefore\ \tan SYP = \frac{p}{d_\phi p}.$$

$$\text{But } \tan SYZ = \frac{p}{d_\phi p} \quad (94);$$

$$\therefore\ SPY = SYZ, \text{ and } SYP,\ SZY \text{ are right angles,}$$

$$\therefore\ PSY = YSZ, \quad \therefore\ \frac{SZ}{SY} = \frac{SY}{SP}.$$

99. Contacts. Let PQ, PR (fig. 8.) be two curves meeting in P; OX, OY rectangular axes. Draw PM, RQN parallel to OY; and let x, y be the co-ordinates of any point in PQ; X, Y those of any point in PR; $MN = h$; y', y'', y''', ...; Y', Y'', Y''', ... the values of $d_x y$, $d_x^2 y$, $d_x^3 y$, ...; $d_X Y$, $d_X^2 Y$, $d_X^3 Y$, ... when x and X are each equal to OM.

Now PM is the value of y and of Y when x and X are each equal to OM; and QN, RN are the values of y, Y respectively, when x and X are each equal to $OM + h$. Therefore

$$QN = PM + y'h + \tfrac{1}{2}y''h^2 + \tfrac{1}{6}y'''h^3 + \ldots$$

$$RN = PM + Y'h + \tfrac{1}{2}Y''h^2 + \tfrac{1}{6}Y'''h^3 + \ldots$$

$$QR = (Y' - y')h + \tfrac{1}{2}(Y'' - y'')h^2 + \tfrac{1}{6}(Y''' - y''')h^3 + \ldots$$

If $Y' = y'$, the curves are said to have a contact of the 1st order: if $Y' = y'$ and $Y'' = y''$, they are said to have a contact of the 2nd order: if $Y' = y'$, $Y'' = y''$, $Y''' = y'''$, they are said to have a contact of the 3d order; and, generally, if the coefficients of h, h^2, h^3, ... h^n in the series for QR, vanish at the same time, the curves are said to have a contact of the n^{th} order.

100. Let D_1, D_2, ... D_n be the values of QR when the curve PR is supposed to have with PQ a contact of the 1st, 2d, ... n^{th} order respectively. The first terms of the series expressing the values of D_1, D_2, ... D_n will involve h, h^2, ... h^n respectively; and, therefore, by diminishing h, any term of the series D_1, D_2, ... D_n may be made less than any of the preceding terms.

Hence, if PR, PS be two curves, of which PS has with PQ a contact of a higher order than PR has, the portion PS of the curve PS lies wholly between PQ and PR, when MN is sufficiently diminished.

101. Let a point n be taken at the distance h to the left of M, and through n suppose a line rqn to be drawn parallel to OY, cutting PQ, PR in q, r. The value of $rn - qn$ may be deduced from the equation

$$RN - QN = (Y' - y')h + \tfrac{1}{2}(Y'' - y'')h^2 + \tfrac{1}{6}(Y''' - y''')h^3 + \ldots$$

by writing $-h$ instead of h;

$$\therefore\ rn - qn = -(Y' - y')h + \tfrac{1}{2}(Y'' - y'')h^2 - \tfrac{1}{6}(Y''' - y''')h^3 + \ldots$$

If the curves have a contact of the 1st order, or $Y' - y' = 0$; $RN - QN$ and $rn - qn$ have the same sign, and the curves touch each other at P but do not intersect; if they have a contact of the 2d order, or $Y' - y' = 0$, $Y'' - y'' = 0$; $RN - QN$ and $rn - qn$ have different signs, and the curves intersect each other in P; and generally, the curves will be found to touch, or cut each other, according as the order of their contact is odd or even.

102. When the equation between X and Y contains one disposable constant, such a value may be given to it, that Y may be equal to y when $X = x$; or, that the curves may meet in the point (x, y).

When the equation contains two disposable constants, such values may be given to them that we may have $Y = y$, $d_X Y = d_x y$, when $X = x$; or, that the curves may have a contact of the 1st order at the point (x, y).

When the equation contains three disposable constants, we may make $Y = y$, $d_X Y = d_x y$, $d_X^2 Y = d_x^2 y$, when $X = x$; or the curves may have a contact of the 2d order at the point (x, y).

In like manner when the equation contains $n + 1$ disposable constants, we may make $Y = y$, $d_X Y = d_x y, \ldots d_X^n Y = d_x^n y$, when $X = x$; or the curves may have a contact of the n^{th} order at the point (x, y).

103. The equation to a straight line is $Y = aX + b$. This contains two constants a, b; therefore a straight line may have a contact of the 1st order with any curve at a point (x, y) in it, when proper values are assigned to a and b. These values are determined by the equations $X = x$, $Y = y$, $d_X Y = d_x y$.

Therefore $y = ax + b$; and $a = d_X Y = d_x y$; therefore the equation becomes $Y - y = d_x y\,(X - x)$. Hence (73) a straight line which has a contact of the 1st order with a curve is a tangent to the curve at the point of contact.

$d_X^2 Y$, $d_X^3 Y$, ... are all equal to nothing. Therefore, if at any point in the curve, $d_x^2 y = 0$, the curve will have a contact of the 2d order with the tangent at that point; if $d_x^3 y = 0$, a contact of the 3d order; and so on. Hence (100), when the first differential coefficient after $d_x y$ that does not vanish, is of an odd order, the curve will have a point of inflexion. Points at which the first differential coefficient after $d_x y$ that does not vanish, is of an even order are called points of undulation.

104. The equation to a circle $c^2 = (Y-b)^2 + (X-a)^2$ contains three constants a, b, c, and therefore a circle may have a contact of the 2d order with any proposed curve at the point (x, y) in it, when proper values are assigned to a, b, c. These values are determined by the equations $X = x$, $Y = y$, $d_X Y = d_x y$, $d_X^2 Y = d_x^2 y$. Hence, (82) a circle which has a contact of the second order with a curve, is the circle of curvature of the curve at the point of contact.

Cor. It will cut the curve (100) at the point (x, y), except $d_X^3 Y = d_x^3 y$.

105. Instead of proving that the straight line and circle which have contacts of the 1st and 2d order with a curve, are the tangent and circle of curvature at the point of contact, we might have made Arts. (99)—(103) precede (73), substituting at the same time the following definitions for those in (73), (82).

A straight line which has a contact of the 1st order with a curve, is called a tangent to the curve at the point of contact.

A circle which has a contact of the 2d order with a curve, is called the circle of curvature at the point of contact.

106. Multiple points. A point in which different branches of a curve intersect or touch each other, is called a double, triple, &c. point, according as the point is common to two, three, &c. branches of the curve.

Let x, y be the co-ordinates of any point in the curve; $u=0$ a rational equation between x and y; a, b the values of x and y at a multiple point; $H, K, L, \dots$; $y', y'', \dots$ the corresponding values of $d_{(x)}u$, $d_{(y)}u$, $d^2_{(x)}u, \dots$; $d_x y$, $d^2_x y, \dots$ If two branches of the curve make a finite angle with each other at the point (a, b), y' will have two values, therefore (65) $H=0$, $K=0$, and $L+2My'+Ny'^2=0$, an equation from which the two values of y' may be deduced.

If three branches of the curve intersect at finite angles in the point (a, b), y' will have three values, therefore (66) $H=0$, $K=0$, $L=0$, and the equation $d^3_x u=0$, after substituting a for x and b for y, will afford a cubic equation from which the values of y' may be determined.

By a similar process we may determine the values of y' when any number of branches intersect.

If several branches of a curve touch each other at the point (a, b), y' will have one value, but y'' will have more than one value. Similarly, if the branches have with each other a contact of the n^{th} order, $y^{n\prime}$ is the first of the quantities $y', y'', \dots$ that has more than one value. Substituting a for x and b for y in the equation $d^n_x u=0$, we get $Ky^{n\prime}+\dots=0$. And $y^{n\prime}$ admits of more than one value, therefore (65) $K=0$, therefore $H=0$.

107. Conjugate points. Let a, b be corresponding possible values of x and y; $y', y'', \dots$ the values of $d_x y$, $d^2_x y, \dots$ when $x=a$, and $y=b$; and suppose one or more of the quantities $y', y'', \dots$ to be impossible. Then, when $x=a\pm h$, the corresponding value of y will be $b\pm y'h+\frac{1}{2}y''h^2\pm\frac{1}{6}y'''h^3+\dots$ an impossible quantity. When this happens the point (a, b) is completely detached from the rest of the curve, and is called a conjugate point.

Let $y^{n\prime}$ be the first impossible differential coefficient. Then, substituting a and b for x and y in $d^n_x u=0$, we get $Ky^{n\prime}+\dots=0$. And $y^{n\prime}$ is the only impossible quantity in this equation, therefore $K=0$, therefore $H=0$.

108. Let a and b be values of x and y that satisfy the equations $u = 0$, $d_{(x)}u = 0$, $d_{(y)}u = 0$. Then, if the coefficients of $d_x y$ and $(d_x y)^2$ in the equation $d_x^2 u = 0$, remain finite when a and b are substituted for x and y, the roots of the resulting quadratic being possible and unequal, two branches of the curve intersect at a finite angle in the point (a, b) (104). When the roots are impossible the point (a, b) will be a conjugate point (107). If the coefficients vanish we must substitute a and b for x and y in the equation $d_x^3 u = 0$. If the roots of this equation be real and unequal, three branches of the curve intersect at finite angles. If two of the roots be impossible the curve will not have a multiple point. If all the roots be impossible the curve will have a conjugate point. If the coefficients of $(d_x y)^3$, $(d_x y)^2$, $d_x y$ vanish we must substitute a and b for x and y in $d_x^4 u = 0$, and so on, till we get an equation in which the coefficients of the powers of $d_x y$ are finite. The number of possible unequal roots in this equation will indicate the number of branches of the curve that intersect at finite angles. And if the roots be impossible the curve will have a conjugate point at the point (a, b).

If the equation have any number of equal roots, a like number of branches of the curve will touch each other. In this case the different corresponding values of $d_x^2 y$ may be found by a process similar to that employed in finding the values of $d_x y$. If the roots of the equation for determining the values of $d_x^2 y$ be impossible, the curve will have a conjugate point.

In like manner if $d_x^n y$ be the first differential coefficient which has real unequal values when $x = a$, and $y = b$, the branches of the curve will have a contact of the $(n - 1)^{\text{th}}$ order. And if its values be impossible the curve has a conjugate point.

109. Let x, y be the co-ordinates of any point in a curve; $u = 0$ a rational equation between x and y. If y become infinite when a is substituted for x in the equation $u = 0$, the straight line $x = a$ will coincide with the curve at

an infinite distance from the origin, and will, therefore, be an asymptote to the curve.

In like manner, if the substitution of b for y render x infinite, the straight line $y = b$, will be an asymptote.

110. At a point where a tangent to the curve is parallel to the axis of x, $d_x y = 0$ (73); therefore, since $d_{(x)} u + d_{(y)} u \cdot d_x y = 0$, we must have $d_{(x)} u = 0$, and $d_{(y)} u$ finite, also, at a point where the tangent is perpendicular to the axis of x, $d_x y = \infty$, and, therefore we must have $d_{(x)} u$ finite, and $d_{(y)} u = 0$.

111. To determine the nature of a curve at any point in it.

Let x, y be the co-ordinates of any point in a curve; a, b corresponding values of x and y. Substitute $a + h$ for x in the equation to the curve; and let $b + Ah^{\alpha} + Bh^{\beta} + \dots$ be one of the values of y expressed in a series of ascending powers of h, therefore, if $y = b + k$,

$$k = Ah^{\alpha} + Bh^{\beta} + \dots$$

And when h is sufficiently diminished the sign of k will depend upon that of Ah^{α}.

Let the coefficients A, $B, \dots$ be possible; and let α, $\beta, \dots$ be either whole numbers, or fractions having odd denominators. Therefore k will be possible whatever be the sign of h. First, let α be greater than 1. Then, since $k \div h$ vanishes when $k = 0$, a tangent to the curve at the point (a, b) will be parallel to the axis of x. When the numerator of α is an odd number, the sign of Ah^{α} changes with that of h; therefore the curve will have a point of inflexion as in fig. (9) or (10) according as A is positive or negative.

When the numerator of α is an even number, the sign of Ah^{α} does not change with that of h; therefore the curve resembles fig. (11) or (12), and y is a maximum or a minimum, according as A is negative or positive.

Next, let α be less than 1. In this case $k \div h$ becomes infinite when $h = 0$; therefore a tangent to the curve at the point (a, b)

is perpendicular to the axis of x. Then, when the numerator of α is an odd number the sign of Ah^{α} changes with that of h; therefore the curve has a point of inflexion as in fig. (13) or (14) according as A is positive or negative.

When the numerator of α is an even number, the sign of Ah^{α} does not change with that of h, therefore the curve resembles fig. (15) or (16) according as A is positive or negative.

Lastly, let $\alpha = 1$, $\therefore\ k = Ah + Bh^{\beta} + \dots$

$$d_x k = d_h k \ (37) \ = A + \beta B h^{\beta-1} + \dots$$

$d_{x=a} y = d_{x=a} k = A$, therefore a tangent to the curve at the point (a, b) makes a finite angle with the axis of x.

$$d_x^2 k = d_h^2 k = \beta(\beta - 2)\, B h^{\beta-2} + \dots$$

therefore $d_{x=a}^2 y = d_{x=a}^2 k$ will be infinite, finite or nothing, according as $\beta - 2$ is negative, nothing or positive.

When the numerator of β is an odd number, the numerator of $\beta - 2$ will be an odd number; therefore when the sign of h changes, the sign of $\beta(\beta - 1) B h^{\beta-1}$ will change, and, therefore, that of $d_x^2 y$. Hence, (a, b) will be the co-ordinates of a point of inflexion.

When any of the coefficients A, B, ... are impossible, k will be impossible, except $h = 0$, whatever be the sign of h. And the point (a, b) will be a conjugate point.

112. When any of the exponents α, β, ... have an even denominator, k cannot be possible for both positive and negative values of h.

k may be impossible both when h is positive and when h is negative. In that case the point (a, b) is a conjugate point.

Since any term of the series $Ah^{\alpha} + Bh^{\beta} + \dots$ may be made greater than the sum of all that follow by diminishing h, the two branches which owe their origin to the double value of any term of the series for k, will have their position ultimately determined by the value of the preceding terms of the series.

113. Let α be the only exponent having an even denominator. Therefore k will have two values for each value of h.

A tangent to the curve at the point (a, b) will be parallel or perpendicular to the axis of x according as α is greater or less than 1 (111). Hence, when α is greater than 1 the curve will resemble fig. (17) or (18), and when α is less than 1 it will resemble (19) or (20).

If $\alpha = 1$, and the denominator of β be even, the term Bh^β will have a positive and negative value; therefore the curve will resemble fig. (21) or (22).

If the denominators of $\alpha, \ldots$ be odd, the first term which involves a power of h, the exponent of which has an even denominator, will give rise to two branches nearly coinciding with the branch determined by the preceding terms of the equation $k = Ah^\alpha + Bh^\beta + \ldots$; and the sign of h cannot change. Therefore the curve will assume one of the forms in fig. (23) or (24), according as α is greater or less than 1.

In the same manner it appears that if $\alpha = 1$, the numerator of $\beta, \ldots$ odd, and that of one of the following exponents even, the curve will assume one of the forms in fig. (25).

114. To find the equation to a curve which touches a series of curves described after a given law.

Let PQ, QP' (fig. 26.) be two of the curves intersecting in Q; PP' the touching curve; P, P' its points of contact with PQ, QP'; $u = 0$, $u' = 0$ the equations to PQ, QP'; and let u contain a constant a, which in u' becomes $a + \delta a$, u and u' being the same in all other respects.

Now, if δa be continually diminished, P' approaches P and ultimately coincides with it; therefore also Q ultimately coincides with P; or, the point in which PP' touches PQ, is the point in which PQ is untimately intersected by QP'.

Since u is changed into u' by substituting $a + \delta a$ for a, the equation $u' = 0$ becomes

$$u + d_a u \, . \, \delta a + \tfrac{1}{2} d_a^2 u \, . \, (\delta a)^2 + \ldots = 0.$$

Therefore, observing that $u = 0$, and dividing by δa,

$$d_a u + \tfrac{1}{2} d_a^2 u \,.\, \delta a + \ldots = 0.$$

The co-ordinates of Q, the point in which PQ, QP' intersect, satisfy the equations $u = 0$, $u' = 0$, and, therefore, the equation $d_a u + \frac{1}{2} d_a^2 u \,.\, \delta a + \ldots = 0$. And when δa vanishes, Q coincides with P, therefore the co-ordinates of P satisfy the equations $u = 0$, $d_a u = 0$.

By eliminating a between $u = 0$ and $d_a u = 0$, we obtain an equation between the co-ordinates of P independent of a; or, the equation to the curve PP'.

The quantity a, which remains constant for the same curve, but which changes in passing from one curve to another, is called a parameter.

115. If P, Q be two points in a curve, P' the ultimate intersection of normals to the curve at P and Q; then, when PQ is indefinitely diminished, P' is the centre of curvature of the curve at P.

Let x, y be the co-ordinates of P; X, Y those of any point in the normal at P; therefore (76) the equation to the normal is

$$(Y - y)\, d_x y + X - x = 0,$$

x is here the parameter; therefore (114) the values of X and Y that satisfy the preceding equation and

$$d_x \left\{ (Y - y)\, d_x y + X - x \right\} = 0,$$

$$\text{or } (Y - y)\, d_x^2 y - (d_x y)^2 - 1 = 0,$$

are the co-ordinates of P'. But the values of X and Y that satisfy these equations are the co-ordinates of the centre of curvature (82); therefore P' is the centre of curvature of PQ at P.

EXAMPLES.

1. To find $d_x\,\{a+3bx-cx^2\}$.

$$d_x\,\{a+3bx-cx^2\}=d_x\,\{3bx-cx^2\}\ (6)=d_x(3bx)-d_x(cx^2)\ (7)$$

$$=3b-2cx\ (6)\ (10).$$

2. To find $d_x\left\{a+b\sqrt{x}-\frac{c}{x}\right\}$.

$$d_x\left\{a+b\sqrt{x}-\frac{c}{x}\right\}=d_x\,\{a+bx^{\frac{1}{2}}-cx^{-1}\}=d_x\{bx^{\frac{1}{2}}-cx^{-1}\}\ (6)$$

$$=d_xbx^{\frac{1}{2}}+d_x(-cx^{-1})\ (7)=b\tfrac{1}{2}x^{\frac{1}{2}-1}-c\,(-1)\,x^{-1-1}\ (10)$$

$$=\frac{b}{2\sqrt{x}}+\frac{c}{x^2}.$$

3. To find $d_x\,(a+bx^2)^3$.

Let $z=a+bx^2$, $\therefore\ d_xz=bd_xx^2\ (6)\ =b2x\ (10)$.

$$d_x\,(a+bx^2)^3=d_xz^3=d_zz^3.d_xz\ (5)\ =3z^2.d_xz\ (10)\ =3\,(a+bx^2)^2\,b2x$$
$$=6b\,(a+bx^2)^2\,x.$$

4. To find $d_x\sqrt{(a+bx+cx^2)}$.

Let $z=a+bx+cx^2$, $\therefore\ d_xz=b+2cx$.

$$d_x\sqrt{(a+bx+cx^2)}=d_xz^{\frac{1}{2}}=d_zz^{\frac{1}{2}}.d_xz=\tfrac{1}{2}z^{\frac{1}{2}-1}.d_xz\,\frac{d_xz}{2\sqrt{z}}$$

$$=\frac{b+2cx}{2\sqrt{(a+bx+cx^2)}}.$$

5. To find $d_x\,(a+bx^2)^3\,x^5$.

Let $z=a+bx^2$, $\therefore\ d_xz=2bx$.

$$d_x\,\{(a+bx^2)^3x^5\}=d_x(z^3x^5)=z^3d_xx^5+x^5d_xz^3=z^3d_xx^5+x^5d_zz^3d_xz$$

$$=z^35x^4+x^53z^2d_xz=5\,(a+bx^2)^3\,x^4+6b\,(a+bx^2)^2\,x^6.$$

6. To find $d_x \left\{a - bx^{-\frac{3}{2}} + (c^2 - x^2)^{\frac{5}{2}}\right\}^{\frac{3}{4}}$.

Let $z = c^2 - x^2$; $y = a - bx^{-\frac{3}{2}} + (c^2 - x^2)^{\frac{5}{2}} = a - bx^{-\frac{3}{2}} - z^{\frac{5}{2}}$.

$d_x z = -2x$, $d_x y = -\left(-\frac{3}{2}\right) bx^{-\frac{3}{2}-1} - \frac{5}{2} z^{\frac{5}{2}-1} . d_x z = \frac{3}{2} bx^{-\frac{5}{2}} + 5z^{\frac{3}{2}} . x.$

$$d_x \left\{a - bx^{-\frac{3}{2}} + (c^2 - x^2)^{\frac{5}{2}}\right\}^{\frac{3}{4}} = d_x y^{\frac{3}{4}} = \frac{3}{4} y^{-\frac{1}{4}} d_x y = \frac{3}{4} \frac{\frac{3}{2} bx^{-\frac{5}{2}} + 5z^{\frac{3}{2}} x}{y^{\frac{1}{4}}}$$

$$= \frac{9bx^{-\frac{5}{2}} + 30(c^2 - x^2)^{\frac{3}{2}} x}{4\left\{a - bx^{-\frac{3}{2}} + (c^2 - x^2)^{\frac{5}{2}}\right\}^{\frac{1}{4}}}.$$

7. To find $d_x \left\{(a + bx^2)^3 (c + ex^4)^5\right\}$.

Let $y = a + bx^2$, $z = c + ex^4$;

$\therefore$ $d_x y = 2bx$, $d_x z = 4ex^3$

$$d_x \left\{(a + bx^2)^3 (c + ex^4)^5\right\} = d_x (y^3 z^5) = y^3 d_x z^5 + z^5 d_x y^3$$

$$= y^3 d_z z^5 . d_x z + z^5 d_y y^3 . d_x y = 5y^3 z^4 . d_x z + 3z^5 y^2 d_x y$$

$$= 20(a + bx^2)^3 (c + ex^4)^4 ex^3 + 6(c + ex^4)^5 (a + bx^2)^2 bx.$$

8. To find $d_x \dfrac{x}{\sqrt{(a - x)}}$.

Let $z = a - x$, $\therefore$ $d_x z = -1$.

$$d_x \frac{x}{\sqrt{(a - x)}} = d_x \frac{x}{z^{\frac{1}{2}}} = \frac{1}{(z^{\frac{1}{2}})^2} (z^{\frac{1}{2}} d_x x - x d_x z^{\frac{1}{2}}) \ (15)$$

$$= \frac{1}{z} \left(z^{\frac{1}{2}} - \frac{x}{2 . z^{\frac{1}{2}}} d_x z\right) = \frac{1}{a - x} \left\{\sqrt{(a - x)} + \frac{x}{2\sqrt{(a - x)}}\right\} = \frac{2a - x}{2(a - x)^{\frac{3}{2}}}.$$

9. $d_x \log_e \dfrac{a + x}{a - x} = d_x \log_e (a + x) - d_x \log_e (a - x)$

$$= \frac{d_x (a + x)}{a + x} - \frac{d_x (a - x)}{a - x} \ (22) = \frac{1}{a + x} + \frac{1}{a - x}$$

$$= \frac{2a}{a^2 - x^2}.$$

10. $$d_x \log_\epsilon \{x + \surd(x^2 \pm a^2)\} = \frac{d_x \{x + \surd(x^2 \pm a^2)\}}{x + \surd(x^2 \pm a^2)} \quad (22)$$

$$= \frac{1 + x\,(x^2 \pm a^2)^{-\frac{1}{2}}}{x + \surd(x^2 \pm a^2)} = \frac{1}{\surd(x^2 \pm a^2)}.$$

11. $$d_x \log_\epsilon \frac{x}{a + \surd(a^2 \pm x^2)} = d_x \log_\epsilon x - d_x \log_\epsilon \{a + \surd(a^2 \pm x^2)\}$$

$$= \frac{d_x x}{x} - \frac{d_x \{a + \surd(a^2 \pm x^2)\}}{a + \surd(a^2 \pm x^2)} = \frac{1}{x} - \frac{\pm x\,(a^2 \pm x^2)^{-\frac{1}{2}}}{a + \surd(a^2 \pm x^2)}$$

$$= \frac{a}{x \surd(a^2 \pm x^2)}.$$

12. $$d_x \log_\epsilon (\log_\epsilon x) = \frac{d_x (\log_\epsilon x)}{\log_\epsilon x} = \frac{1}{x \log_\epsilon x}.$$

13. $$d_x (x \log_a x) = x\, d_x \log_a x + \log_a x = \frac{1}{\log_\epsilon a} + \log_a x.$$

14. $$d_x \log_\epsilon \cos x = \frac{d_x \cos x}{\cos x} = \frac{-\sin x}{\cos x} \; (29) = -\tan x.$$

15. $$d_x \operatorname{ver\,sin}^{-1} \frac{2x}{a} = \frac{d_x \dfrac{2x}{a}}{\surd\left\{2 . \dfrac{2x}{a} - \left(\dfrac{2x}{a}\right)^2\right\}} \quad (30)$$

$$= \frac{1}{\surd(ax - x^2)}.$$

16. $$d_x \log_\epsilon \tan x = \frac{d_x \tan x}{\tan x} = \frac{(\text{secant } x)^2}{\tan x} = \frac{2}{\sin 2x}.$$

17. $$d_x \log_\epsilon (\cos x + \sqrt{-1} \sin x) = \frac{d_x (\cos x + \sqrt{-1} \sin x)}{\cos x + \sqrt{-1} \sin x}$$

$$= \frac{-\sin x + \sqrt{-1} \cos x}{\cos x + \sqrt{-1} \sin x} = \sqrt{-1}.$$

18. $$d_x \tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan x\right\}$$

$$= \frac{d_x\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan x\right\}}{1+\frac{a-b}{a+b}(\tan x)^2} \qquad (30)$$

$$= \frac{\sqrt{\left(\frac{a-b}{a+b}\right)}(\text{secant } x)^2}{1+\frac{a-b}{a+b}(\tan x)^2} = \frac{\sqrt{(a^2-b^2)}}{(a+b)(\cos x)^2+(a-b)(\sin x)^2}$$

$$= \frac{\sqrt{(a^2-b^2)}}{a+b\cos 2x}.$$

19. To expand $(a+bx+cx^2+dx^3+\dots)^n$ in a series of ascending powers of x.

Let $(a+bx+cx^2+dx^3+\dots)^n = A+Bx+Cx^2+Dx^3+\dots$; therefore, differentiating the logarithms of these quantities,

$$n\,\frac{b+2cx+3dx^2+\dots}{a+bx+cx^2+dx^3+\dots} = \frac{B+2Cx+3Dx^2+\dots}{A+Bx+Cx^2+Dx^3+\dots};$$

therefore, multiplying both sides by the product of the denominators, and arranging the terms according to powers of x,

$$\begin{aligned}
&nbA+2ncAx+3ndAx^2+4neAx^3+\dots\\
&\qquad+nbBx+2ncBx^2+3ndBx^3+\dots\\
&\qquad\qquad+nbCx^2+2ncCx^3+\dots\\
&\qquad\qquad\qquad+nbDx^3+\dots\\
&\qquad\qquad\qquad\qquad+\dots\\
&=aB+2aCx+3aDx^2+4aEx^3+\dots\\
&\qquad+bBx+2bCx^2+3bDx^3+\dots\\
&\qquad\qquad+cBx^2+2cCx^3+\dots\\
&\qquad\qquad\qquad+dBx^3+\dots\\
&\qquad\qquad\qquad\qquad+\dots
\end{aligned}$$

Therefore, equating the coefficients of like powers of x,

$$aB = nbA,$$

$$2aC + bB = 2ncA + nbB,$$

$$3aD + 2bC + cB = 3ndA + 2ncB + nbC,$$

&c. = &c.

Also, making $x = 0$ in the equation

$$(a + bx + \dots)^n = A + Bx + \dots$$

we get $A = a^n$;

$$\therefore\ B = \frac{n}{1} a^{n-1} b,$$

$$C = na^{n-1} + \frac{n}{1} \frac{n-1}{2} a^{n-2} b,$$

$$D = na^{n-1} d + \frac{n}{1} \frac{n-1}{1} a^{n-2} bc + \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3} a^{n-3} b^3,$$

&c. = &c.

$$\therefore\ (a + bx + cx^2 + dx^3 + \dots)^n$$

$$= a^n + \frac{n}{1} a^{n-1} bx + \left(\frac{n}{1} a^{n-1} c + \frac{n}{1} \frac{n-1}{2} a^{n-2} b\right) x^2$$

$$+ (na^{n-1} d + \frac{n}{1} \frac{n-1}{1} a^{n-2} bc + \frac{n}{1} \frac{n-1}{2} \frac{n-2}{3} a^{n-3} b^3)\, x^3 + \dots$$

20. To expand tan x in a series of ascending powers of x.

Tan $(-x) = -\tan x$, therefore the series for tan x contains no even powers of x. Also $\{\tan x\}_{x=0} = 0$. Therefore we may assume

$$\tan x = ax + bx^3 + cx^5 + dx^7 + ex^9 + \dots$$

$$\therefore\ d_x \tan x = a + 3bx^2 + 5cx^4 + 7dx^6 + 9ex^8 + \dots$$

But $d_x \tan x = 1 + (\tan x)^2$

$$= 1 + a^2x^2 + 2abx^4 + b^2x^6 + 2adx^8 + \dots$$

$$+ 2acx^6 + 2bcx^8 + \dots$$

therefore, equating the coefficients of like powers of x,

$$a = 1;\quad b = \tfrac{1}{3}a^2 = \tfrac{1}{3};\quad c = \tfrac{1}{5}2ab = \frac{2}{3.5};$$

$$d = \tfrac{1}{7}(b^2 + 2ac) = \frac{17}{3.3.5.7};\quad e = \tfrac{1}{9}(2ad + 2bc)\frac{62}{3.3.5.7.9};$$

$$\therefore\ \tan x = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3.3.5.7} + \frac{62x^9}{3.3.5.7.9} + \dots$$

21. To expand $\sin(x+h)$ and $\cos(x+h)$ by Taylor's Theorem.

$$\left.\begin{aligned} \sin(x+h) &= \sin x + d_x \sin x\,\frac{h}{1} + d_x^2 \sin x\,\frac{h^2}{1.2} + d_x^3 \sin x\,\frac{h^3}{1.2.3} + \dots \\ \cos(x+h) &= \cos x + d_x \cos x\,\frac{h}{1} + d_x^2 \cos x\,\frac{h^2}{1.2} + d_x^3 \cos x\,\frac{h^3}{1.2.3} + \dots \end{aligned}\right\}\ (38)$$

therefore, substituting the values of $d_x \sin x$, $d_x^2 \sin x, \dots$ $d_x \cos x$, $d_x^2 \cos x, \dots$ (32), we get

$$\sin(x+h) = \sin x + \cos x\,\frac{h}{1} - \sin x\,\frac{h^2}{1.2} - \cos x\,\frac{h^3}{1.2.3} + \dots$$

$$\cos(x+h) = \cos x - \sin x\,\frac{h}{1} - \cos x\,\frac{h^2}{1.2} + \sin x\,\frac{h^3}{1.2.3} + \dots$$

22. To expand $\tan^{-1}(x+h)$.

$$d_x \tan^{-1} x = \frac{1}{1+x^2}\quad (33);$$

$$\therefore\ d_x^2 \tan^{-1} x = d_x \frac{1}{1+x^2} = -\frac{2x}{(1+x^2)^2};$$

$$d_x^3 \tan^{-1} x = d_x \left\{-\frac{2x}{(1+x^2)^2}\right\} = \frac{2(3x^2-1)}{(1+x^2)^3}; \ldots$$

$$\therefore\ \tan^{-1}(x+h) = \tan^{-1} x + \frac{1}{1+x^2}\,\frac{h}{1} - \frac{2x}{(1+x^2)^2}\,\frac{h^2}{1\,.\,2}$$

$$+ \frac{2(3x^2-1)}{(1+x^2)^3}\,\frac{h^3}{1\,.\,2\,.\,3} + \ldots$$

If $\tan^{-1} x = \dfrac{\pi}{2} - z$, $\cot z = x$, $\therefore\ d_x z = -(\sin z)^2$.

$$d_x \tan^{-1} x = -d_x z = (\sin z)^2,$$

$$d_x^2 \tan^{-1} x = \quad d_x z\,.\,2 \sin z \cos z = -(\sin z)^2 \sin 2z,$$

$$\frac{1}{1\,.\,2}\,d_x^3 \tan^{-1} x = -d_x z\,\{\sin 2\,.\,\cos z\,.\,\sin 2z + (\sin z)^2 \cos 2z\}$$

$$= -d_x z\,.\,\sin z\,.\,\sin 3z = (\sin z)^3 \sin 3z,$$

$$\frac{1}{1\,.\,2\,.\,3}\,d_x^4 \tan^{-1} x = \quad d_x z\,\{(\sin z)^2 \cos z \sin 3z + (\sin z)^2 \cos 3z\}$$

$$= \quad d_x z\,(\sin z)^2\,.\,\sin 4z = -(\sin z)^4 \sin 4z,$$

&c. = &c.

$$\therefore\ \tan^{-1}(x+h) = \tan^{-1} x + \sin z\,.\,\sin z\,\frac{h}{1} - (\sin z)^2 \sin 2z\,\frac{h^2}{2}$$

$$+ (\sin z)^3 \sin 3z\,\frac{h^3}{3} - \ldots - (-1)^n (\sin z)^n \sin nz\,\frac{h^n}{n} - \ldots$$

23. To find $d_x^n \{a+bx+cx^2\}^r$.

$d_x^n \{a+bx+cx^2\}^n = \underline{|n}$. coefficient of h^n in the expansion of

$$\{a+b(x+h)+c(x+h)^2\}^r \quad (45).$$

If $a+bx+cx^2=p$, $b+2cx=q$,

$$\begin{aligned}
\frac{1}{\underline{|r}}\{a+b(x+h)+c(x+h)^2\}^r &= \frac{1}{\underline{|r}}\{p+qh+ch^2\}^r \\
&= \frac{1}{\underline{|r}}(p+qh)^r \\
&+ \frac{1}{\underline{|r-1}}\,\frac{1}{\underline{|1}}(p+qh)^{r-1}\,c\,h^2 \\
&+ \frac{1}{\underline{|r-2}}\,\frac{1}{\underline{|2}}(p+qh)^{r-2}\,c^2\,h^4 \\
&+ \frac{1}{\underline{|r-3}}\,\frac{1}{\underline{|3}}(p+qh)^{r-3}\,c^3\,h^6 \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$\begin{aligned}
= \quad & \ldots + \frac{1}{\underline{|r-n}}\,\frac{1}{\underline{|n}}\,p^{r-n}\,.\,q^n\,.\,h^n + \ldots \\
+ \frac{1}{\underline{|1}} & \left\{\ldots + \frac{1}{\underline{|r-n+1}}\,\frac{1}{\underline{|n-2}}\,p^{r-n+1}\,.\,q^{n-2}\,.\,h^{n-2} + \ldots\right\} c\,h^2 \\
+ \frac{1}{\underline{|2}} & \left\{\ldots + \frac{1}{\underline{|r-n+2}}\,\frac{1}{\underline{|n-4}}\,p^{r-n+2}\,.\,q^{n-4}\,.\,h^{n-4} + \ldots\right\} c^2\,h^4 \\
+ \frac{1}{\underline{|3}} & \left\{\ldots + \frac{1}{\underline{|r-n+3}}\,\frac{1}{\underline{|n-6}}\,p^{r-n+3}\,.\,q^{n-6}\,.\,h^{n-6} + \ldots\right\} c^3\,h^6; \\
+ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$\therefore\ d_x^n\{a+bx+cx^2\}$$

$$\begin{aligned}
= \frac{\underline{|r}}{\underline{|n}} & \left\{\frac{1}{\underline{|r-n}}\,\frac{1}{\underline{|n}}\,p^{r-n}q^n + \frac{1}{\underline{|r-n+1}}\,\frac{1}{\underline{|n-2}}\,p^{r-n+1}q^{n-2}c\right. \\
& \left. + \frac{1}{\underline{|r-n+2}}\,\frac{1}{\underline{|n-4}}\,p^{r-n+2}q^{n-4}c^2 + \frac{1}{\underline{|r-n+3}}\,\frac{1}{\underline{|n-6}}\,p^{r-n+3}q^{n-6}c^3 + \ldots\right\}.
\end{aligned}$$

24. To find the first $(n+1)$ terms of the expansion of $\log_\epsilon (x+h)$, and the remainder.

$$f(x+h)=f(x)+d_x f(x)\frac{h}{\underline{|1}}+d_x^2 f(x)\frac{h^2}{\underline{|2}}+\ldots$$

$$+d_x^n f(x)\frac{h^n}{\underline{|n}}+d_x^n P\frac{h^{n+1}}{\underline{|n}} \quad (41)$$

$$\therefore \text{ if } f(x)=\log_\epsilon x,\ d_x f(x)=\frac{1}{x},\ d_x^2 f(x)=-\frac{1}{x^2},\ d_x^3 f(x)=\frac{2}{x^3},\ldots$$

$$d_x^n \log_\epsilon x=-(-1)^n\frac{\underline{|n-1}}{x^n} \quad (23)$$

$$P=\{f(x+h)-f(x)\}\frac{1}{h}=\{\log_\epsilon (x+h)-\log_\epsilon x\}\frac{1}{h}$$

$$d_x^n P=-(-1)^n\frac{\underline{|n-1}}{h}\left\{\frac{1}{(x+h)^n}-\frac{1}{x^n}\right\}.$$

$$\therefore \log_\epsilon (x+h)=\log_\epsilon x+\frac{h}{x}-\frac{h^2}{2x^2}+\ldots$$

$$-(-1)^n\frac{h^n}{nx^n}-(-1)^n\frac{h^n}{n}\left\{\frac{1}{(x+h)^n}-\frac{1}{x^n}\right\}.$$

25 If $f(x)=x^m$.

$$d_x^n f(x)=\frac{\underline{|m}}{\underline{|m-n}}x^{m-n}. \quad (12).$$

$$P=\{f(x+h)-f(x)\}\frac{1}{h}=\{(x+h)^m-x^m\}\frac{1}{h}.$$

$$d_x^n P=\frac{\underline{|m}}{\underline{|m-n}}\{(x+h)^{m-n}-x^{m-n}\}\frac{1}{h}.$$

$$(x+h)^m = x^m + \frac{m}{1}x^{m-1}h + \frac{m}{1}\,\frac{m-1}{2}\,x^{m-2}h^2 + \dots$$

$$+\frac{\underline{\lvert m}}{\underline{\lvert m-n}}\,x^{m-n}\,\frac{h^n}{\underline{\lvert n}} + \frac{\underline{\lvert m}}{\underline{\lvert m-n}}\left\{(x+h)^{m-n} - x^{m-n}\right\}\frac{h^n}{\underline{\lvert n}}.$$

$$\therefore\ \frac{(x+h)^m}{\underline{\lvert m}} = \frac{x^m}{\underline{\lvert m}} + \frac{x^{m-1}}{\underline{\lvert m-1}}\,\frac{h}{1} + \frac{x^{m-2}}{\underline{\lvert m-2}}\,\frac{h^2}{\underline{\lvert 2}} + \dots$$

$$+\frac{x^{m-n+1}}{\underline{\lvert m-n+1}}\,\frac{h^{n-1}}{\underline{\lvert n-1}} + \frac{(x+h)^{m-n}}{\underline{\lvert m-n}}\,\frac{h^n}{\underline{\lvert n}}.$$

26. To expand sec x in a series of ascending powers of x by Maclaurin's Theorem.

$$\sec x = \{\sec x\}_{x=0} + d_{x=0}\sec x\,\frac{x}{1} + d^2_{x=0}\sec x\,\frac{x^2}{1\,.\,2}$$

$$+\,d^3_{x=0}\sec x\,\frac{x^3}{1\,.\,2\,.\,3} + d^4_{x=0}\sec x\,\frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \dots \quad (39).$$

$d_x \sec x = \tan x\,.\sec x \quad (30),$

$d_x^2 \sec x = \sec x + 2\,(\tan x)^2 \sec x,$

$d_x^3 \sec x = 5\tan x\,.\sec x - 6\,(\tan x)^3 \sec x,$

$d_x^4 \sec x = 5\,(\sec x)^3 - 2\,(\tan x)^2 \sec x - 24\,(\tan x)^4 \sec x,$

&c. = &c.

$\{\sec x\}_{x=0} = 1,$

$d_{x=0}\sec x = 0,$

$d^2_{x=0}\sec x = 1,$

$d^3_{x=0}\sec x = 0,$

$d^4_{x=0}\sec x = 5,$

&c. = &c.

$$\therefore\ \sec x = 1 + \frac{x^2}{1\,.\,2} + \frac{5x^4}{1\,.\,2\,.\,3\,.\,4} + \dots$$

Cotan x and cosec x involve a negative power of x, and versin^{-1}x involves fractional powers of x; therefore (40) they cannot be expanded by Maclaurin's Theorem.

27. To expand y in a series of ascending powers of x, x and y being connected by the equation $y^3 - axy - b^3 = 0$.

Differentiating with respect to x, we get

$$(3y^2 - ax)\, d_x y - ay = 0,$$

$$(3y^2 - ax)\, d_x^2 y + 6y\,(d_x y)^2 - 2a\,d_x y = 0,$$

$$(3y^2 - ax)\, d_x^3 y + 3\,(6y\,d_x y - a)\, d_x^2 y + 6(d_x y)^3 = 0.$$

....

$$\therefore\; y_{x=0} = b,\;\; d_{x=0} y = \frac{a}{3b},\;\; d_{x=0}^2 y = 0,\;\; d_{x=0}^3 y = -\frac{2a^3}{27b^5},\ldots$$

$$\therefore\; y = b + \frac{ax}{3b} - \frac{a^3x^3}{81b^5} + \ldots$$

28. To expand y^n in terms of x and z by Lagrange's Theorem; x, y, z being connected by the equation $y = z + xy^r$.

Comparing this equation with (50) we get

$$\phi(y) = y^r,\;\; f(y) = y^n;$$

$$\therefore\; \phi(z) = z^r,\;\; f(z) = z^n.$$

$$d_z f(z) = nz^{n-1},\;\; \phi(z)\, d_z f(z) = nz^{n+r-1}.$$

$$\overline{\phi(z)}|^2\, d_z f(z) = nz^{n+2r-1},$$

$$d_z\left\{\overline{\phi(z)}|^2\, d_z f(z)\right\} = n(n+2r-1)z^{n+2r-2}.$$

$$\overline{\phi(z)}|^3\, d_z f(z) = nz^{n+3r-1},$$

$$d_z^2\left\{\overline{\phi(z)}|^2\, d_z f(z)\right\} = n\,(n+3r-1)\,(n+3r-2)\,z^{n+3r-3}.$$

Therefore substituting these values of $\phi(z)$, $d_z f(z)$, ... in (50) we get

$$y^n = z^n + \frac{n}{1} z^{n+r-1} x + \frac{n}{1} \frac{n+2r-1}{2} z^{n+2r-2} x^2$$

$$+ \frac{n}{1} \frac{n+3r-1}{2} \frac{n+3r-2}{3} z^{n+3r-3} x^3 + \ldots$$

29. Reversion of series. If

$$0 = a + by + cy^2 + dy^3 + \ldots;$$

to find the value of y.

Let $z = -\frac{a}{b}$, $\therefore\ y = z - \left(\frac{c}{b} y^2 + \frac{d}{b} y^3 + \ldots\right)$,

comparing this with $y = z + x\phi(y)$ we get

$$x = -1,\ \phi(y) = \frac{c}{b} y^2 + \frac{d}{b} y^3 + \frac{e}{b} y^4 + \ldots;$$

$$\therefore\ \phi(z) = \frac{c}{b} z^2 + \frac{d}{b} z^3 + \frac{e}{b} z^4 + \ldots$$

$$\left.\overline{\phi(z)}\right|^2 = \frac{c^2}{b^2} z^4 + 2\frac{cd}{b^2} z^5 + \ldots,$$

$$d_z \left\{\left.\overline{\phi(z)}\right|^2\right\} = 4\frac{c^2}{b^2} z^3 + 10\frac{cd}{b^2} z^4 + \ldots$$

$$\left.\overline{\phi(z)}\right|^3 = \frac{c^3}{b^3} z^6 + 3\frac{c^2 d}{b^3} z^7 + \ldots,$$

$$d_z^2 \left\{\left.\overline{\phi(z)}\right|^3\right\} = 6.5\frac{c^3}{b^3} z^4 + 7.6.3\frac{c^2 d}{b^3} z^5 + \ldots$$

&c. = &c.

$$\therefore\ (50)\quad y = z - \left(\frac{c}{b}z^2 + \frac{d}{b}z^3 + \frac{e}{b}z^4 + \dots\right)$$

$$+ \left(2\frac{c^2}{b^2}z^3 + 5\frac{c\,d}{b^2}z^4 + \dots\right) - \left(5\frac{c^3}{b^3}z^4 + \dots\right) + \dots$$

$$= -\frac{a}{b} - \left(\frac{ca^2}{b^3} - \frac{da^3}{b^4} + \frac{ea^4}{b^5} + \dots\right)$$

$$+ \left(-2\frac{c^2a^3}{b^5} + 5\frac{c\,d\,a^4}{b^6} + \dots\right) - \left(5\frac{c^3a^4}{b^7} + \dots\right).$$

30. Let $a + h$ be one of the roots of the equation

$$\phi(x) = 0;\ \{\phi(x)\}_{x=a} = u,\ d_{x=a}\phi(x) = u',\ d^2_{x=a}\phi(x) = u'', \dots$$

$$\therefore\ 0 = u + u'h + \tfrac{1}{2}u''h^2 + \tfrac{1}{6}u'''h^3 + \tfrac{1}{24}u''''h^4 + \dots$$

$$\therefore\ h = -\frac{u}{u'} - \left\{\tfrac{1}{2}\frac{u''}{u'}h^2 + \tfrac{1}{6}\frac{u'''}{u'}h^3 + \tfrac{1}{24}\frac{u''''}{u'}h^4 + \dots\right\}.$$

Comparing this with the equation $y = z + x\phi(y)$ we get

$$y = h,\quad z = -\frac{u}{u'},\quad x = -1,$$

$$\phi(z) = \tfrac{1}{2}\frac{u''}{u'}z^2 + \tfrac{1}{6}\frac{u'''}{u'}z^3 + \tfrac{1}{24}\frac{u''''}{u'}z^4 + \dots$$

$$d_z\left.\overline{\phi(z)}\right|^2 = \frac{(u'')^2}{(u')^2}z^3 + \tfrac{5}{6} - \frac{u''\,u'''}{(u')^3}z^4 + \dots$$

$$d^2_z\left.\overline{\phi(z)}\right|^3 = \frac{6\,.\,5}{8}\,\frac{(u'')^3}{(u')^3}z^4 + \dots$$

&c. = &c.

$$\therefore\ h = -\frac{1}{u'}u - \tfrac{1}{2}\frac{u''}{(u')^3}u^2 + \tfrac{1}{6}\frac{u'''}{(u')^4}u^3 - \tfrac{1}{24}\frac{u''''}{(u')^5}u^4 + \ldots$$

$$- \tfrac{1}{2}\frac{(u'')^2}{(u')^5}u^3 + \tfrac{5}{12}\frac{u''u'''}{(u')^6}u^4 + \ldots$$

$$- \tfrac{5}{8}\frac{(u'')^3}{(u')^7}u^4 + \ldots$$

If a be a near approximation to one of the roots of the equation $\phi(x) = 0$, u will be very small, and the series expressing the value of h will converge rapidly.

31. To differentiate y^z; y and z being functions of x.

$$d_x y^z = d_{(y)} y^z \,.\, d_x y + d_{(z)} y^z \,.\, d_x z \quad (52)$$

$$= z y^{z-1} \,.\, d_x y + \log_e y \,.\, y^z \,.\, d_x z.$$

32. To differentiate v^{y^z}; v, y, z being functions of x.

$$\text{Let } y^z = t, \ \therefore\ v^{y^z} = v^t.$$

$$d_x v^{y^z} = d_x v^t = d_{(v)} v^t \,.\, d_x v + d_{(t)} v^t \,.\, d_x t$$

$$= t v^{t-1} \,.\, d_x v + \log_e v \,.\, v^t \,.\, d_x t.$$

$$d_x t = d_x y^z = z y^{z-1} \,.\, d_x y + \log_e y \,.\, y^z \,.\, d_x z;$$

$$\therefore\ d_x v^{y^z} = y^z \,.\, \frac{v^{y^z}}{v} d_x v + \log_e v \,.\, v^{y^z} (z y^{z-1} d_x y + \log_e y \,.\, y^z \,.\, d_x z).$$

33. To find $d_x y$, $d_x^2 y, \ldots$; where x and y are connected by the equation $y^3 - 3axy + x^3 = 0$.

Let $u = y^3 - 3axy + x^3$, $\therefore\ u = 0$, $\therefore\ d_{(x)} u + d_{(y)} u . d_x y = 0$, and $d_{(x)}^2 u + 2 d_{(x)} d_{(y)} u . d_x y + d_{(y)}^2 u . (d_x y)^2 + d_{(y)} u . d_x^2 y = 0.$

$$d_{(x)} u = -3ay + 3x^2,\quad d_{(y)} u = 3y^2 - 3ax;$$

$$\therefore\ x^2 - ay + (y^2 - ax)\, d_x y = 0.$$

$$d^2_{(x)} u = 3.2x,\quad d_{(x)} d_{(y)} u = -3a,\quad d^2_{(y)} u = 3.2y;$$

$$\therefore\ 2x - 2a d_x y + 2y (d_x y)^2 + (y^2 - ax)\, d^2_x y = 0.$$

34. If $x^4 + 2ax^2 y - by^3 = 0.$

$$d_{(x)} u = 4x^3 + 4axy,\quad d_{(y)} u = 2ax - 3by^2;$$

$$\therefore\ 4x^3 + 4axy + (2ax - 3by^2)\, d_x y = 0.$$

$$d^2_{(x)} u = 1.2x^2 + 4ay,\quad d_{(x)} d_{(y)} u = 4ax,\quad d^2_{(y)} u = -6by;$$

$$\therefore\ 1.2x^2 + 4ay + 8ax . d_x y - 6by\, (d_x y)^2 + (2ax - 3by^2)\, d^3_x y = 0.$$

35. To eliminate a from the equation

$$ay^2 - bx^2 + 2cxy = 0.$$

$$-bx + cy + (ay + cx) d_x y = 0,\quad \therefore\ ay d_x y = bx - cy - cx d_x y;$$

therefore substituting this value of a in the original equation

$$bxy - cy^2 + (cxy - bx^2)\, d_x y = 0.$$

Eliminating b we get

$$ay^2 + cxy + (axy + cx^2)\, d_x y = 0.$$

36. To eliminate a from the equation

$$2ay - x^2 + a^2 = 0.$$

$$-x + a d_x y = 0,\quad \therefore\ a = \frac{x}{d_x y};$$

$$\therefore\ 2y d_x y - x (d_x y)^2 + x = 0.$$

37. To eliminate a and c from the equation

$$y - c\epsilon^{-x^3} - a = 0.$$

$$d_x y + 3cx^2 \epsilon^{-x^3} = 0,$$

$$d^2_x y + 6cx\, \epsilon^{-x^3} - 9cx^4\, \epsilon^{-x^3} = 0.$$

Whence we obtain

$$x d_x^2 y + (3x^3 - 2) d_x y = 0.$$

38. To eliminate ϵ^x from the equation

$$(\epsilon^{2x} - 1) u = \epsilon^{2x} + 1.$$

$$(\epsilon^{2x} - 1) d_x u + 2u\epsilon^{2x} = 2\epsilon^{2x},$$

$$\therefore\ \epsilon^{2x}(d_x u + 2u - 2) = d_x u;$$

$$\therefore\ 1 - u^2 = d_x u.$$

39. To eliminate ϵ^x and $\cos x$ from the equation

$$u = \epsilon^x . \cos x.$$

$$d_x u = \epsilon^x \cos x - \epsilon^x \sin x,$$

$$d_x^2 u = -2\epsilon^x \sin x;$$

$$\therefore\ 2d_x u = 2u + d_x^2 u.$$

40. To eliminate $\phi(x^2 - y^2)$ from the equation

$$z = (x+y)^n \phi(x^2 - y^2).$$

Let $x^2 - y^2 = r$; $\therefore\ d_x r = 2x,\ \ d_y r = -2y$.

$$z = (x+y)^n \phi(r) \quad (a)$$

$$\begin{aligned} d_x z &= (x+y)^n d_x \phi(r) + n(x+y)^{n-1} \phi(r) \\ &= (x+y)^n d_r \phi(r)\, d_x r + n(x+y)^{n-1} \phi(r) \\ &= 2x(x+y)^n d_r \phi(r) + n(x+y)^{n-1} \phi(r) \quad (\beta). \end{aligned}$$

$$\begin{aligned} d_y z &= (x+y)^n d_y \phi(r) + n(x+y)^{n-1} \phi(r) \\ &= (x+y)^n d_r \phi(r)\, d_y r + n(x+y)^{n-1} \phi(r) \\ &= -2y(x+y)\, d_r \phi(r) + n(x+y)^{n-1} \phi(r) \quad (\gamma). \end{aligned}$$

Therefore, eliminating $\phi(r)$ and $d_r \phi(r)$ between (a), (β), (γ), we get

$$y d_x z + x d_y z = nz.$$

41. To eliminate $F(x+at)$, $f(x-at)$ from the equation

$$y = F(x+at) + f(x-at).$$

$$\text{Let } x + at = r, \quad x - at = s;$$

$$\therefore \; d_x r = 1, \quad d_t r = a, \quad d_s x = 1, \quad d_t s = -a.$$

$$y = F(r) + f(s);$$

$$\begin{aligned} d_x y = d_x F(r) + d_x f(s) &= d_r F(r)\, d_x r + d_s f(s)\, d_x s \\ &= d_r F(r) + d_s f(s). \end{aligned}$$

$$\begin{aligned} d_x^2 y = d_x d_r F(r) + d_x d_r f(s) &= d_r^2 F(r)\, d_x r + d_s^2 f(s)\, d_x s \\ &= d_r^2 F(r) + d_s^2 f(s). \end{aligned}$$

$$\begin{aligned} d_t y = d_t F(r) + d_t f(s) &= d_r F(r)\, d_t r + d_s f(s)\, d_t s \\ &= a d_r F(r) - a d_s f(s). \end{aligned}$$

$$\begin{aligned} d_t^2 y = d_t \{a d_r F(r)\} - d_t \{a d_s f(s)\} &= a d_r^2 F(r)\, d_t r - a d_s^2 f(s)\, d_t s \\ &= a^2 d_r^2 F(r) + a^2 d_s^2 f(s). \end{aligned}$$

$$\therefore \; d_t^2 y = a^2 d_x^2 y.$$

42. To find the value of $\dfrac{x - x^{n+1}}{1 - x^2}$, when $x = 1$.

$$\text{Let } u = x - x^{n+1}, \quad v = 1 - x^2.$$

$$u_{x=1} = 0, \quad v_{x=1} = 0.$$

$$d_x u = 1 - (2n+1)\, x^{2n}, \quad d_{x=1} u = -2n.$$

$$d_x v = -2x, \quad d_{x=1} v = -2.$$

$$\therefore \; (62) \; \frac{u_{x=1}}{v_{x=1}} = \frac{d_{x=1} u}{d_{x=1} v} = \frac{-2n}{-2} = n.$$

43. To find the value of $\dfrac{2x^3-3x^2-12x+20}{x^3-x^2-8x+12}$, when $x=2$.

Let $u=2x^3-3x^2-12x+20$, $v=x^3-x^2-8x+12$.

$$u_{x=2}=0,\quad v_{x=2}=0.$$

$$d_xu=6x^2-6x-12,\quad d_{x=2}u=0.$$

$$d_xv=3x^2-2x-8,\quad d_{x=2}v=0.$$

$$d_x^2u=12x-6,\quad d_{x=2}^2u=18.$$

$$d_x^2v=6x-2,\quad d_{x=2}^2u=10.$$

$$\therefore\ (62)\ \frac{u_{x=2}}{v_{x=2}}=\frac{d_{x=2}^2u}{d_{x=2}^2u}=\frac{18}{10}.$$

44. To find the value of $\dfrac{(x^2-a^2)^{\frac{3}{2}}}{(x^3-a^3)^{\frac{3}{2}}}$; when $x=a$.

In this case, if u and v denote the numerator and denominator of the fraction respectively, u, v, d_xu, d_xv vanish, and d_x^2u, d_x^2v become infinite, when $x=a$. We must, therefore, substitute $a+h$ for x in u and v, and expand (63).

$$u_{x=a+h}=(2ah+h^2)^{\frac{3}{2}}=(2ah)^{\frac{3}{2}}\left\{1+\tfrac{3}{4}\frac{h}{a}+\dots\right\}.$$

$$v_{x=a+h}=(3a^2h+3ah^2+\dots)^{\frac{3}{2}}=(3a^2h)^{\frac{3}{2}}\left\{1+\tfrac{3}{2}\frac{h}{a}+\dots\right\}.$$

$$\therefore\ \frac{u_{x=a+h}}{v_{x=a+h}}=\frac{(2ah)^{\frac{3}{2}}\left\{1+\frac{3}{4}\frac{h}{a}+\dots\right\}}{(3a^2h)^{\frac{3}{2}}\left\{1+\frac{3}{2}\frac{h}{a}+\dots\right\}}=\left(\frac{2}{3a}\right)^{\frac{3}{2}}\frac{1+\frac{3}{4}\frac{h}{a}+\dots}{1+\frac{3}{2}\frac{h}{a}+\dots}.$$

$$\therefore\ \frac{u_{x=a}}{v_{x=a}}=\left(\frac{2}{3a}\right)^{\frac{3}{2}}.$$

45. To find the maxima and minimum values of

$$x^5 - 5x^4 + 5x^3 + 1.$$

$$\text{Let } u = x^5 - 5x^4 + 5x^3 + 1\,;$$

$$\therefore\ d_x u = 5x^4 - 5.4x^3 + 5.3x^2.$$

The values of x that render u a maximum or a minimum are found among those that satisfy the equation $d_x u = 0$ (68).

The roots of $d_x u = 0$, or of $5x^4 - 5.4x^3 + 5.3x^2 = 0$ are 3, 1, 0.

$d_x^2 u = 5\,(4x^3 - 12x^2 + 6x)$.

$d_{x=3}^2 u = 5.18$, a positive quantity;

$$\therefore\ u_{x=3} = -26, \text{ a minimum.}$$

$$d_{x=1}^2 u = -5.2 \text{ a negative quantity;}$$

$$\therefore\ u_{x=1} = 2, \text{ a maximum.}$$

$d_{x=0}^2 u = 0$, $\therefore$ $u_{x=0}$ is neither a maximum nor a minimum.

46. If $y^3 - 3axy + x^3 = 0$, to determine the maximum values of y.

Differentiating by the rule in (53) we get

$$(y^2 - ax)\,d_x y + x^2 - ay = 0,$$

$$(y^2 - ax)\,d_x^2 y + 2y\,(d_x y)^2 - 2a\,d_x y + 2x = 0.$$

When y is a maximum or a minimum $d_x y = 0$, $\therefore$ $x^2 - ay = 0$, therefore substituting this value of y in the original equation, $x^6 - 2a^3x^3 = 0$, the values of x are $2^{\frac{1}{3}}a$, 0.

When $x = 2^{\frac{1}{3}}a$, $y^3 - 2\,.\,2^{\frac{1}{3}}a^2 y = 0$.

When $x = 2^{\frac{1}{3}}a$, $y^3 - 2^{\frac{4}{3}}a^2 y = 0$,

$$\therefore\ y = 2^{\frac{2}{3}}a, \text{ or } 0.$$

When $y = 2^{\frac{2}{3}}a$, $y^2 - ax$ is positive, therefore $d_x^2 y$ is negative, therefore $2^{\frac{2}{3}}a$ is a maximum value of y.

When $x = 0$, we have $y = 0$, $d_x y = \frac{0}{0}$, $d_x^2 y = \frac{0}{0}$.

In this case (65) $d_x y = 0$, $d_x^2 y = \frac{2}{3a}$. Therefore 0 is a minimum value of y.

47. If $z = y^3 + x^3 - 3axy$; to find the maximum and minimum values of z.

$$d_x z = 3(x^2 - ay),$$

$$d_y z = 3(y^2 - ax),$$

$$d_x^2 z = 6x,\ d_x d_y z = -3a,\ d_y^2 z = 6y.$$

The values of x and y that make $d_x z = 0$, $d_y z = 0$, are $x = a$, $y = a$, and $x = 0$, $y = 0$.

$$d_{x=a}^2 z \,.\, d_{y=a}^2 z = 36a^2,\ (d_{x=a} d_{y=a} z)^2 = 9a^2,$$

$$d_{x=a}^2 z \,.\, d_{y=a}^2 z - (d_{x=a} d_{y=a} z)^2 = 27a^2 \text{ a positive quantity.}$$

And $d_{x=a}^2 z$ is positive; therefore (71) $-a^3$ is a minimum value of z.

$$d_{x=0}^2 z = 0,\ d_{y=0}^2 z = 0,\ d_{x=0} d_{y=0} z = -3a;$$

$$\therefore\ d_{x=0}^2 z \,.\, d_{y=0}^2 z - (d_{x=0} d_{y=0} z)^2 = -9a^2.$$

Therefore when $x = 0$, $y = 0$, z is neither a maximum nor a minimum.

48. If $xyz = a^3$, and $u = 2xy + 2xz + 2yz$; to find the values of x and y that make u a minimum.

Since $xyz = a^3$, z is a function of x and y; therefore differentiating the equation $xyz = a^3$, first with respect to x, and then with respect to y, we get

$$yz + xy d_x z = 0,$$

$$xz + xy d_y z = 0.$$

And $d_x u = 2y + 2z + 2x d_x z + 2y d_x z$,

$$d_y u = 2x + 2x d_y z + 2z + 2z + 2y d_y z.$$

Making $d_x u = 0$, $d_y u = 0$, and substituting the values of $d_x z$, $d_y z$, found above, we obtain

$$x - z = 0, \quad y - z = 0;$$

$$\therefore \; x = y = z = a.$$

And the minimum value of u is $6a^2$.

49. If $a^x b^y c^z = K$, and $u = (x+1)(y+1)(z+1)$; to find the values of x and y that render u a maximum.

$a^x b^y c^z = K$, $\therefore$ $x \log_\epsilon a + y \log_\epsilon b + z \log_\epsilon c = \log_\epsilon K$; z is a function of x and y; therefore, differentiating, we get

$$\log_\epsilon a + \log_\epsilon c \,.\, d_x z = 0, \quad \log_\epsilon b + \log_\epsilon c \,.\, d_y z = 0.$$

$$\frac{d_x u}{u} = \frac{1}{x+1} + \frac{d_x z}{z+1}, \quad \frac{d^y u}{u} = \frac{1}{y+1} + \frac{d_y z}{z+1}.$$

Making $d_x u = 0$, and substituting the values of $d_x z$, $d_y z$ found above, we get

$$(x+1) \log_\epsilon a = (y+1) \log_\epsilon b = (z+1) \log_\epsilon c.$$

$$\therefore \; x = \frac{1}{3 \log_\epsilon a} \left\{ \log_\epsilon (bcK) - 2 \log_\epsilon a \right\},$$

$$y = \frac{1}{3 \log_\epsilon b} \left\{ \log_\epsilon (acK) - 2 \log_\epsilon b \right\},$$

$$z = \frac{1}{3 \log_\epsilon c} \left\{ \log_\epsilon (abK) - 2 \log_\epsilon c \right\};$$

$$u = \frac{1}{3^3} \frac{\left\{ \log_\epsilon (abcK) \right\}^3}{\log_\epsilon a \,.\, \log_\epsilon b \,.\, \log_\epsilon c}.$$

50. The equation to the ellipse is

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1; \quad \therefore \quad \frac{y d_x y}{b^2} + \frac{x}{a^2} = 0.$$

The equation to a tangent to a curve at the point (x, y) is

$$Y - y = d_x y (X - x).$$

Substituting the value of $d_x y$ given above in this equation, we get for the equation to a tangent to the ellipse

$$Y - y = -\frac{b^2}{a^2}\frac{x}{y}(X - x);$$

therefore reducing, and observing that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$\frac{Yy}{b^2} + \frac{Xx}{a^2} = 1.$$

The subtangent $= -\frac{y}{d_x y} = \frac{a^2}{b^2}\frac{y^2}{x}$.

The equation to a normal at the point (x, y) is

$$(Y - y)\, d_x y + X - x = 0. \quad (16).$$

Substituting the value of $d_x y$ in terms of x and y, this becomes

$$b^2 \frac{X}{x} - a^2 \frac{Y}{y} = b^2 - a^2.$$

The subnormal $= y d_x y = -\frac{b^2}{a^2} x$.

The length of the portion of the normal contained between the curve and the axis of x

$$= \sqrt{\{(y d_x y)^2 + y^2\}} = \frac{1}{a^2}\{a^4 y^2 + b^4 x^2\}^{\frac{1}{2}}.$$

Differentiating the equation $\frac{y d_x y}{b^2} + \frac{x}{a^2} = 0$, we get

$$\frac{y d_x^2 y}{b^2} + \frac{(d_x y)^2}{b^2} + \frac{1}{a^2} = 0.$$

$$\therefore\ d_x^2 y = -\frac{b^2}{y}\left\{\frac{(d_x y)^2}{b^2} + \frac{1}{a^2}\right\} = -\frac{b^2}{a^2 y^3}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = -\frac{b^2}{a^2 y^3}.$$

$$1 + (d_x^2 y) = \frac{a^4 y^2 + b^4 x^2}{a^4 y^2}.$$

$$r = -\frac{1}{d_x^2 y}\left\{1 + (d_x y)^2\right\}^{\frac{3}{2}} = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4}$$

$$= \frac{a^2}{b^4}\,(\text{normal})^3.$$

Let X, Y be the co-ordinates of the centre of curvature; then (82)

$$Y = y + \frac{1}{d_x^2 y}\left\{1 + (d_x y)^2\right\} = -(a^2 - b^2)\frac{y^3}{b^4},$$

$$X = x - \frac{d_x y}{d_x^2 y}\left\{1 + (d_x y)^2\right\} = (a^2 - b^2)\frac{x^3}{a^4};$$

$$\therefore\ (aX)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}\frac{x^2}{a^2},$$

$$(bY)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}\frac{y^2}{b^2}.$$

Therefore, adding and observing that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$(aX)^{\frac{2}{3}} + (bY)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

the equation to the evolute (fig. 27.)

51. The equation to the cycloid is

$$y = a \operatorname{ver\,sin}^{-1}\frac{x}{a} + \sqrt{(2ax - x^2)};$$

$$\therefore\ d_x y = \sqrt{\left(\frac{2a - x}{x}\right)},$$

$$d_x^2 y = -\frac{a}{x\sqrt{(2ax - x^2)}}.$$

Let X, Y be the co-ordinates of the centre of curvature at the point x, y; then

$$X = x - \frac{d_x y}{d_x^2 y}\left\{1 + (d_x y)^2\right\} = 4a - x,$$

$$Y = y + \frac{1}{d_x^2 y}\left\{1 + (d_x y)^2\right\}$$

$$= a \text{ ver sin}^{-1}\frac{x}{a} - \sqrt{(2ax - x^2)}$$

$$= a\left\{\pi - \text{ver sin}^{-1}\frac{2a - x}{a}\right\} - \sqrt{(2ax - x^2)};$$

therefore, substituting the value of x in terms of X, we get for the equation to the evolute

$$a\pi - Y = a \text{ ver sin}^{-1}\frac{X - 2a}{a} + \sqrt{\left\{2a(X - 2a) - (X - 2a)^2\right\}}.$$

This is the equation to an equal cycloid (fig. 28); its axis is parallel to the axis of x, and the co-ordinates of its vertex are $x = 2a$, $y = a\pi$.

52. To draw an asymptote to the curve

$$y^3 - 3axy + x^3 = 0. \quad \text{(fig. 29).}$$

$$-\left(\frac{y}{x}\right)^3 = 1 - 3\frac{a}{x}\frac{y}{x};$$

$$\therefore\ -\frac{y}{x} = \left(1 - 3\frac{a}{x}\frac{y}{x}\right)^{\frac{1}{3}}$$

$$= 1 - \frac{a}{x}\frac{y}{x} - \frac{a^2}{x^2}\left(\frac{y}{x}\right)^2 - \dots$$

$$= 1 - \frac{a}{x}\left(\overline{-1 - \frac{a}{x}\frac{y}{x} - \dots}\right) - \frac{a^2}{x^2}\left(\frac{y}{x}\right)^2 - \dots$$

$$= 1 + \frac{a}{x} + \frac{a}{x^2}\frac{y}{x} - \dots - \frac{a^2}{x^2}\left(\frac{y}{x}\right)^2 - \dots$$

$$\therefore\ 0 = y + x + a - \text{terms in } \frac{1}{x},\ \frac{1}{x^2},\ + \ldots$$

therefore (75) $y + x + a = 0$ is the equation to the asymptote. Hence if we take OT, OV each equal to a, OX, OY being the axes to which the curve is referred, TV will be the asymptote.

The derived equation to the curve is

$$(y^2 - ax)\, d_x y + x^2 - ay = 0.$$

When $d_x y = 0$, $x^2 = ay$, $\therefore x = 2^{\frac{1}{3}}a$, $y = 2^{\frac{2}{3}}a$. These values of x and y do not cause $y^2 - ax$ to vanish; $\therefore$ (109) a tangent to the curve at the point $x = 2^{\frac{1}{3}}a$, $y = 2^{\frac{2}{3}}a$, will be parallel to the axis of x.

In the same manner it appears that a tangent to the curve at the point $x = 2^{\frac{2}{3}}a$, $y = 2^{\frac{1}{3}}a$, is perpendicular to the axis of x.

If we suppose x^3 to vanish, we have $y^3 - 3axy = 0$; $\therefore$ $y^2 = 3ax$. If we substitute this value of x in x^3, the dimensions of y will be higher in this term than in either of the others. Consequently x^3 vanishes compared with the other terms when x is very small. Hence a portion of the curve near the origin will have for its approximate equation $y^2 = 3ax$. This portion will resemble *boc* (112). In the same manner we shall find that y^3 vanishes compared with the other terms when y is very small, and that, consequently, $x^2 = 3ay$ is an approximate equation to the portion *doe* of the curve.

53. Let the equation to a curve be

$$y^3 - 2xy^2 + x^2y - a^3 = 0;$$

x becomes infinite when $y = 0$, therefore (108) the axis of x is an asymptote.

$$\text{Also } \frac{y}{x}\left\{\left(\frac{y}{x}\right)^2 - 2\frac{y}{x} + 1\right\} = \frac{a^3}{x^3};$$

$$\therefore\ \frac{y}{x}-1=\left(\frac{a}{x}\right)^{\frac{3}{2}}\div\left(\frac{y}{x}\right)^{\frac{1}{2}}=\left(\frac{a}{x}\right)^{\frac{3}{2}}\div\left\{1+\left(\frac{a}{x}\right)^{\frac{3}{2}}\right\}^{\frac{1}{2}}$$

$$=\left(\frac{a}{x}\right)^{\frac{3}{2}}\times\left\{1-\frac{1}{2}\left(\frac{a}{x}\right)^{\frac{3}{2}}\right\}$$

$$=\left(\frac{a}{x}\right)^{\frac{3}{2}}-\frac{1}{2}\left(\frac{a}{x}\right)^{3}+\dots$$

$$\therefore\ y=x+\frac{a^{\frac{3}{2}}}{x^{\frac{1}{2}}}-\dots$$

therefore $y=x$ is the equation to an asymptote.

54. Let the equation to the curve be

$$y^5-2a^2x^2y+x^5=0\ \text{(fig. 30.)}$$

$$\therefore\ -\left(\frac{y}{x}\right)^5=1-2\left(\frac{a}{x}\right)^2\frac{y}{x}:$$

$$\therefore\ -\frac{y}{x}=1-\frac{2}{5}\left(\frac{a}{x}\right)^2\frac{y}{x}-\dots$$

$$=1+\frac{2}{5}\left(\frac{a}{x}\right)^2-\dots\ \text{substituting the value of } \frac{y}{x}:$$

$$\therefore\ 0=y+x+\frac{2}{5}\left(\frac{a}{x}\right)^2-\dots$$

$\therefore$ $y+x=0$ is the equation to an asymptote.

x^5 will be found to vanish, compared with the other terms, when x is very small. The approximate equation in this case is $y^5-2a^2x^2y=0$. This gives $y^2=\sqrt{2}\,a\,x$, $y^2=-\sqrt{2}\,a\,x$. These are the approximate equations to the portions *bc*, *de* of the curve.

y^5 will be found to vanish compared with the other terms when y is very small. The approximate equation in this case is $x^5 - 2a^2x^2y = 0$, $\therefore\ x^3 = 2a^2y$. This is the approximate equation to the portion fg.

55. Let the equation to the curve be

$$xy^2 - ey = ax^3 + bx^2 + cx + d.$$

$$\therefore\ \left(\frac{y}{x}\right)^2 = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \frac{e}{x^2}\,\frac{y}{x};$$

$$\therefore\ \pm\frac{y}{x} = \sqrt{\left(a + \frac{b}{x} + \ldots\right)}\ \sqrt{\left\{1 + \frac{e}{x^2}\,\frac{y}{x}\,\frac{1}{a + \frac{b}{x} + \ldots}\right\}}$$

$$= \sqrt{a}\left\{1 + \frac{b}{2ax} + \ldots\right\}\left\{1 + \frac{1}{2}\,\frac{e}{x^2}\,\frac{y}{x}\,\frac{1}{a + \frac{b}{x} + \ldots}\right\}$$

$$= \sqrt{a} + \frac{b}{2\sqrt{a}\,.\,x} + \text{terms in } \frac{1}{x^2},\ \frac{1}{x^3},\ \ldots$$

$$\therefore\ y = \sqrt{a}\,x + \frac{b}{2\sqrt{a}} + \ldots$$

$$-y = \sqrt{a}\,x + \frac{b}{2\sqrt{a}} + \ldots$$

$$\therefore\ y = \sqrt{a}\,x + \frac{b}{2\sqrt{a}},$$

$$-y = \sqrt{a}\,x + \frac{b}{2\sqrt{a}},$$

are the equations to asymptotes.

Also y is infinite when $x = 0$; therefore the axis of y will be an asymptote.

56. Let the equation to the curve be

$$y^4 - 2x^2y^2 - x^4 + 2axy^2 - 5ax^3 = 0.$$

$$\therefore\ \left(\frac{y}{x}\right)^4 - 2\left(\frac{y}{x}\right)^2 + 1 = 2 + \left\{5 - 2\left(\frac{y}{x}\right)^2\right\}\frac{a}{x};$$

$$\therefore\ \left(\frac{y}{x}\right)^2 - 1 = \sqrt{2} + \frac{1}{2\sqrt{2}}\left\{5 - 2\left(\frac{y}{x}\right)^2\right\}\frac{a}{x} + \dots$$

$$\therefore\ \left(\frac{y}{x}\right)^2 = 1 + \sqrt{2} + \frac{1}{2\sqrt{2}}\left\{5 - 2\left(\frac{y}{x}\right)^2\right\}\frac{a}{x} + \dots$$

$$= 1 + \sqrt{2} + \frac{1}{2\sqrt{2}}\left\{3 - 2\sqrt{2} + \dots\right\}\frac{a}{x} + \dots$$

$$\therefore\ \frac{y}{x} = \pm\left\{\sqrt{(1 + \sqrt{2})} + \frac{3 - 2\sqrt{2} + \dots}{8\sqrt{(2 + 2\sqrt{2})}}\,\frac{a}{x} + \dots\right\};$$

$$\therefore\ y = \sqrt{(1 + \sqrt{2})}\,x + \frac{3 - 2\sqrt{2}}{8\sqrt{(2 + 2\sqrt{2})}}\,a,$$

$$y = -\sqrt{(1 + \sqrt{2})}\,x - \frac{3 - 2\sqrt{2}}{8\sqrt{(2 + 2\sqrt{2})}}\,a,$$

are the equations to the asymptotes.

In all these cases y might have been expanded in the required form by Lagrange's Theorem.

57. Let $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$ be the equation to a curve (fig. 31.)

The derived equation is

$$3a(y^2 + ay)\,d_xy + 2(a^2x - x^3) = 0.$$

The pairs of values of x and y that satisfy the equations $y^2 + ay = 0$, $a^2x - x = 0$, and the equation to the curve are

$$x = a,\ y = 0;\quad x = -a,\ y = 0;\quad x = 0,\ y = -a.$$

The second derived equation is

$$3a(y^2+ay)\,d_x^2y + 3a(2y+a)\,(d_xy)^2 + 2a^2 - 6x^2 = 0.$$

Therefore, y' denoting the value of d_xy when $x = a$, $y = 0$,

$$3y'^2 = 4, \quad \therefore\ y' = \pm\sqrt{\tfrac{4}{3}}.$$

And the values of y are possible when $x = a \pm h$, h being small; therefore $x = a$, $y = 0$, are the co-ordinates of a double point. In like manner we find that $x = -a$, $y = 0$, are the co-ordinates of a multiple point, at which the values of d_xy are $\pm\sqrt{\frac{4}{3}}$: and also that $x = 0$, $y = -a$, are the co-ordinates of a multiple point at which the values of d_xy are $\pm\sqrt{\frac{2}{3}}$.

58. $\dfrac{\rho}{a} = \dfrac{\theta^2}{\theta^2 - 1}$ is the equation to a spiral (fig. 32).

When $\rho = a$, $\theta = \infty$; therefore the spiral will have an asymptotic circle the radius of which is a. ρ becomes infinite when $\theta^2 - 1 = 0$, or when $\theta = 1$ or -1. Hence the spiral will have two asymptotes.

$$\text{If } u = \frac{1}{\rho}, \quad au = 1 - \frac{1}{\theta^2}; \quad \therefore\ ad_\theta u = \frac{2}{\theta^3};$$

$$\frac{1}{p^2} = u^2 + (d_\theta u)^2 \ (89) = \frac{1}{a^2}\left\{1 - \frac{2}{\theta^2} + \frac{1}{\theta^4} + \frac{4}{\theta^6}\right\}.$$

$$\text{And when } \theta^2 - 1 = 0, \ \frac{1}{p^2} = \frac{4}{a^2}, \quad \therefore\ p = \frac{a}{2}.$$

Hence if two straight lines be drawn at the distance $\frac{a}{2}$ from the focus, making with SA, the line from which θ is measured, the angles 1 and -1, they will be asymptotes to the spiral.

When $\theta^2 - 3 = 0$, $\rho = \frac{3}{2}a$, and $u + d_\theta^2 u = 0$, while $u^2 + (d_\theta u)^2$ remains finite; therefore the radius of curvature becomes infinite, and the spiral has a point of inflexion.

59. If the circle *QPE* (fig. 33.) touching the circle *BQA* internally, roll upon *BQA*, the curve *BP* traced out by any point *P* in the circumference of *QPE*, is called a hypocycloid. Let *S* be the centre of *BQA*. Through *S* and the centre of *QPE* draw *SEQ*; therefore the circles touch each other in *Q*. The motion of *T* is manifestly perpendicular to *PQ*; therefore *PQ* is a normal, and *PE* a tangent to *BP* at *P*. Draw *SY* perpendicular to *PE*. Let *BP′* be the evolute of *BP*, *P′* the centre of curvature of *BP* at *P*; then (86) *QP* will be a tangent to the evolute at *P′*, and *YP* will be equal to the perpendicular from δ on *PQ*. Let $SQ = a$, $QE = 2b$, $SP = \rho$, $SY = p$, $SP' = \rho'$, $YP = p'$.

$$\frac{PY^2}{QS^2} = \frac{EY^2}{ES^2}, \quad \therefore \frac{1}{a^2}(\rho^2 - p^2) = 1 - \frac{p^2}{(a-2b)^2};$$

$$\therefore \frac{1}{a^2}(\rho d_p \rho - p) = -\frac{p}{(a-2b)^2}.$$

But (96) $\rho'^2 - p'^2 = \rho d_p \rho - p$, and $p'^2 = \rho^2 - p^2$;

$$\therefore \frac{a^4}{(a-2b)^2}(\rho'^2 - p'^2) = 1 - \frac{p'^2}{a^2};$$

or, if $\dfrac{a^2}{a-2b} = a'$ and $a = a' - 2b'$,

$$\frac{1}{a'^2}(\rho'^2 - p'^2) = 1 - \frac{p'^2}{(a'-2b')^2},$$

the polar equation to a hypocycloid, in which the radius of the fixed circle is a', and that of the rolling circle b'.

$$\frac{a'}{a} = \frac{a}{a-2b} = \frac{a'-2b'}{a-2b} = \frac{b'}{b};$$

therefore the hypocycloids *BP*, *BP′* are similar.

When $p = 0$ we have $\rho' = p' = \rho = a$; therefore the apse of $B\rho'$ touches the circle *BQA* at the point where it meets the cusp of *BP*.

60. The extremities of the straight line AB (fig. 34.) move in the rectangular axes OX, OY; to find the curve which AB perpetually touches.

Let $AB = c$, $OA = a$; then the equation to AB will be

$$\frac{x}{a} + \frac{y}{\sqrt{(c^2 - a^2)}} = 1.$$

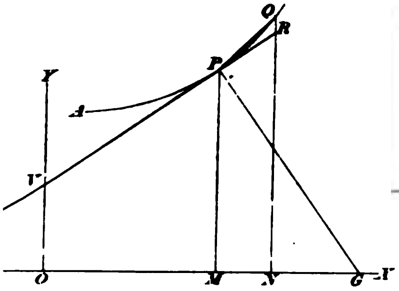
Differentiating this equation with respect to the parameter a (113), we get

$$-\frac{x}{a^2} + \frac{ya}{(c^2 - a^2)^{\frac{3}{2}}} = 0,$$

therefore, eliminating a between this equation and the equation to AB, we obtain $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ the equation to the curve which AB perpetually touches.

ERRATA.

PAGE	LINE	ERROR.	CORRECTION.
4	19	a_n	u_n
8	4	$n-m$	$n-m+1$
12	2	(16)	(17)
16	6	$\sin x > x$	$\sin x < x$
29	17	$d_{x=0}^{n} f(x)$	$d_{x=a}^{n} f(x)$
31	4	$d_x^n P \frac{(z-x)^n}{\underline{\lfloor n}}$	$d_x^n P \frac{(z-x)^{n+1}}{\underline{\lfloor n}}$
31	7	$d_x^n P \frac{h^n}{\underline{\lfloor n}}$	$d_x^n P \frac{h^{n+1}}{\underline{\lfloor n}}$
45	5	$d_y^2 \frac{h^2}{1.2}$	$d_x^2 y \frac{h^2}{1.2}$
50	8	$d_t^2 y (d_t x)^2$	$d_x^2 y (d_t x)^2$
79	21	$\rho\, d_\rho\, p$	$\rho\, d_\rho \rho$
79	23	$\rho\, d_\rho\, p$	$\rho\, d_\rho \rho$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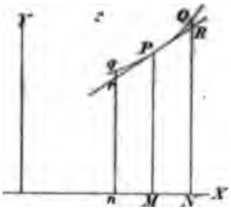
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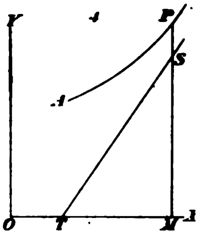
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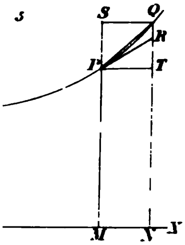
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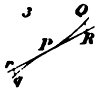
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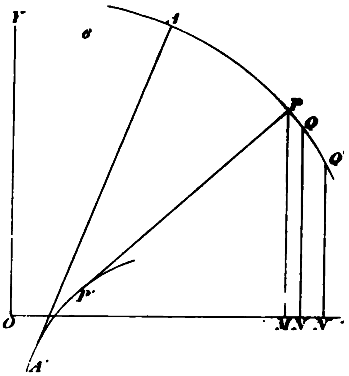
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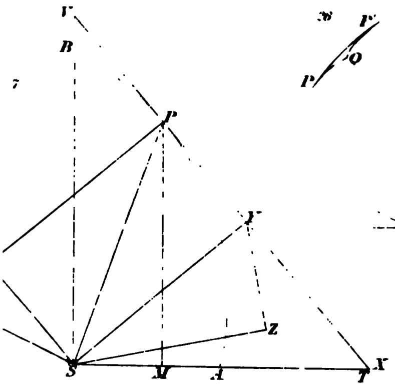
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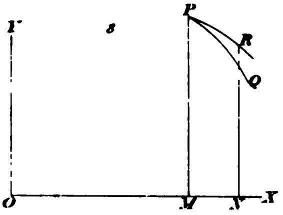
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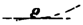
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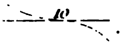
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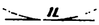
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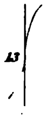
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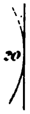
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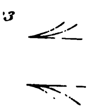
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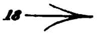
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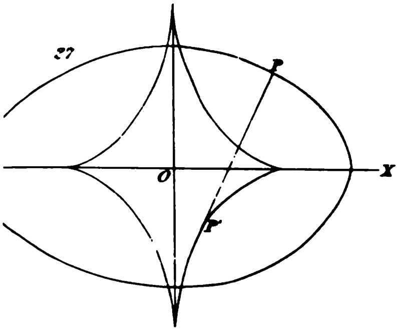

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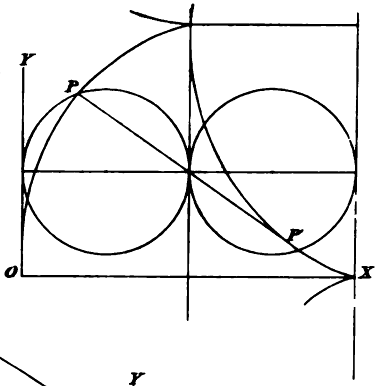

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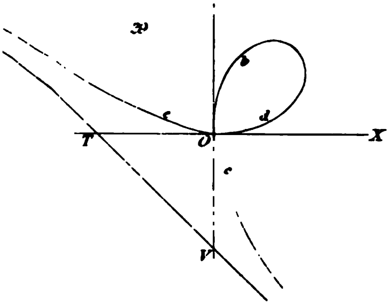

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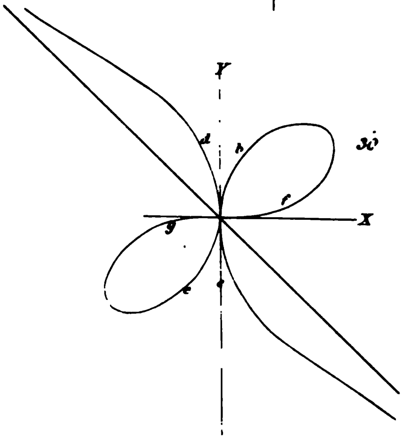

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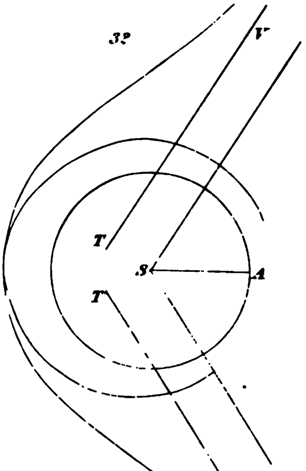

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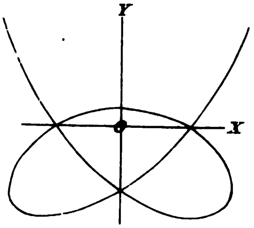

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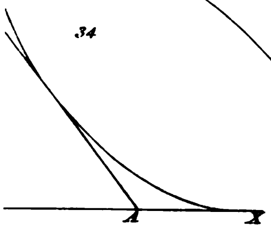

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